FREE BERTINI'S THEOREM AND APPLICATIONS

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Abstract. The simplest version of Bertini's irreducibility theorem states that the generic fiber of a non-composite polynomial function is an irreducible hypersurface. The main result of this paper is its analog for a free algebra: if f is a noncommutative polynomial such that $f-\lambda$ factors for infinitely many scalars λ, then there exist a noncommutative polynomial h and a nonconstant univariate polynomial p such that $f=p\circ h$. Two applications of free Bertini's theorem for matrix evaluations of noncommutative polynomials are given. An *eigenlevel set* of f is the set of all matrix tuples X where $f(X)$ attains some given eigenvalue. It is shown that eigenlevel sets of f and g coincide if and only if $fa=ag$ for some nonzero noncommutative polynomial a. The second application pertains quasiconvexity and describes polynomials f such that the connected component of $\{X \text{ tuple of symmetric } n\times n \text{ matrices}\colon \lambda I\succ f(X)\}$ about the origin is convex for all natural n and $\lambda>0$. It is shown that such a polynomial is either everywhere negative semidefinite or the composition of a univariate and a convex quadratic polynomial.

1. Introduction

Bertini's irreducibility theorem (see e.g. [Sha94, Theorem 2.26]) is a fundamental result with a rich history [Kle98] and ubiquitous in algebraic geometry. When applied to a multivariate polynomial function f over a an algebraically closed field, it states that the hypersurface $\{f=\lambda\}$ is irreducible for all but finitely many values λ unless f is a composite with a univariate polynomial. This particular case is significant in its own right in commutative algebra, and has been extensively studied and generalized [Sch00, BDN09]. In this paper we prove its analog for a free associative algebra and derive consequences of interest for free analysis [K-VV14] and free real algebraic geometry [HKM12, BPT13].

Let $\Bbbk$ be a field and $d\in\mathbb{N}$. Let $\Bbbk\langle\underline{x}\rangle$ be the free associative $\Bbbk$-algebra in freely noncommuting variables $\underline{x}=(x_1,\dots,x_d)$. Its elements are called **noncommutative polynomials**. We say that f **factors** in $\Bbbk\langle\underline{x}\rangle$ if $f=f_1f_2$ for some nonconstant $f_1,f_2\in\Bbbk\langle\underline{x}\rangle$. Otherwise, f is **irreducible** over $\Bbbk$. A nonconstant $f\in\Bbbk\langle\underline{x}\rangle$ is **composite** (over $\Bbbk$) if there exist $h\in\Bbbk\langle\underline{x}\rangle$ and a univariate polynomial $p\in\Bbbk[t]$ such that $\deg p>1$ and $f=p\circ h=p(h)$. Our first main result is the free algebra analog of a special case of the classical Bertini's (irreducibility) theorem.

Theorem A (Free Bertini's theorem)**.** *If $f\in\Bbbk\langle\underline{x}\rangle\setminus\Bbbk$ is not composite, then $f-\lambda$ is irreducible over $\overline{\Bbbk}$ for all but finitely many $\lambda\in\overline{\Bbbk}$.*

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See Theorem 3.2 for a more comprehensive statement and proof. Next we apply Theorem A to matrix evaluations of noncommutative polynomials. Let $f \in \Bbbk<\underline{x}>$. Given $X \in \mathrm{M}_n(\Bbbk)^d$ let $f(X) \in \mathrm{M}_n(\Bbbk)$ be the evaluation of f at X. The **eigenlevel set** of f at $\lambda \in \Bbbk$ is

$$L_\lambda(f) = \bigcup_{n\in\mathbb{N}} \left\{X \in \mathrm{M}_n(\Bbbk)^d \colon \lambda \text{ is an eigenvalue of } f(X)\right\}.$$

In terms of [KV17, HKV18, HKV], eigenlevel sets are free loci of polynomials $\lambda - f$, which have been intensively studied for their implications to domains of noncommutative rational functions [K-VV09], factorization in a free algebra [HKV18, HKV] and matrix convexity [BPT13, HKM13, DD-OSS17]. Using Theorem A we derive the following algebraic certificate for inclusion of eigenlevel sets (see Theorem 4.3 for the proof).

Theorem B. *Let $\Bbbk$ be an algebraically closed field of characteristic 0 and $f, g \in \Bbbk<\underline{x}>$. Then eigenlevel sets of f are contained in eigenlevel sets of g if and only if there exist nonzero $a, h \in \Bbbk<\underline{x}>$ and $p \in \Bbbk[t]$ such that $g = p(h)$ and $fa = ah$.*

Lastly we turn to noncommutative polynomials describing convex matricial sets. Let $\mathrm{S}_n(\mathbb{R}) \subseteq \mathrm{M}_n(\mathbb{R})$ denote the subspace of symmetric matrices. In [BM14], a symmetric $f \in \mathbb{R}<\underline{x}>$ with $f(0) = 0$ is called *quasiconvex* if for every $n \in \mathbb{N}$ and positive definite $A \in \mathrm{S}_n(\mathbb{R})$, the set

$$\{X \in \mathrm{S}_n(\mathbb{R})^d \colon A - f(X) \text{ is positive semidefinite}\} \tag{1.1}$$

is convex; see [BM14, Subsection 1.1] for the relation with the classical (commutative) notion of quasiconvexity. Furthermore, in [BM14, Theorem 1.1] the authors showed that every quasiconvex polynomial is either convex quadratic or minus a sum of hermitian squares (i.e., $-f = \sum_m h_m^* h_m$ for some $h_m \in \mathbb{R}<\underline{x}>$, in which case the set (1.1) equals $\mathrm{S}_n(\mathbb{R})^d$ for every $A \succ 0$ and $n \in \mathbb{N}$).

To relate quasiconvexity more closely to the notion of a free semialgebraic set [HM12, HKM12] in free real algebraic geometry, we say that a symmetric $f \in \mathbb{R}<\underline{x}>$ with $f(0) = 0$ is **locally quasiconvex** if there exists $\varepsilon > 0$ such that the connected component of

$$\{X \in \mathrm{S}_n(\mathbb{R})^d \colon \lambda I - f(X) \text{ is positive definite}\}$$

containing the origin is convex for every $n \in \mathbb{N}$ and $\lambda \in (0, \varepsilon)$.

Theorem C. *If $f \in \mathbb{R}<\underline{x}>$ is locally quasiconvex, then either $-f$ is a sum of hermitian squares or $f = p(\ell_0 + \ell_1^2 + \cdots + \ell_m^2)$ for some $p \in \mathbb{R}[t]$ and linear $\ell_0, \dots, \ell_m \in \mathbb{R}<\underline{x}>$.*

A precise biconditional statement is given in Theorem 5.4 below.

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2. Preliminaries

We start with reviewing certain notions and technical results from the factorization theory of P. M. Cohn [Coh06] that will be used throughout the paper. Most of this theory is based on the fact that $\Bbbk\langle\underline{x}\rangle$ is a free ideal ring (see e.g. [Coh06, Corollary 2.5.2]), which will be implicitly used when referring to the existing literature.

Noncommutative polynomials $f_1, f_2 \in \Bbbk\langle\underline{x}\rangle$ are **stably associated** [Coh06, Section 0.5] if there exist $P_1, P_2 \in \mathrm{GL}_2(\Bbbk\langle\underline{x}\rangle)$ such that $f_2 \oplus 1 = P_1(f_1 \oplus 1)P_2$. Equivalently [Coh06, Proposition 0.5.6 and Theorem 2.3.7], there exist $g_1, g_2 \in \Bbbk\langle\underline{x}\rangle$ such that f_1, g_2 are left coprime, g_1, f_2 are right coprime, and

$$f_1 g_1 = g_2 f_2. \tag{2.1}$$

Here left (right) coprime refers to the absence of a non-invertible common left (right) factor. The importance of stable association stems from the fact that factorization of a noncommutative polynomial into irreducible factors is unique up to stable association of factors [Coh06, Proposition 3.2.9]. The following finiteness result was first proved by G. M. Bergman in his doctoral thesis.

Proposition 2.1 (Bergman, [Coh06, Exercise 2.8.8])**.** *Given $f \in \Bbbk\langle\underline{x}\rangle$, there are (up to a scalar multiple) only finitely many polynomials stably associated to f.*

We will also require degree bounds on "witnesses" of stable association in (2.1).

Lemma 2.2. *If $f_1, f_2 \in \Bbbk\langle\underline{x}\rangle \setminus \Bbbk$ are stably associated, then $\deg f_1 = \deg f_2$ and there exist nonzero $g_1, g_2 \in \Bbbk\langle\underline{x}\rangle$ such that $\deg g_i < \deg f_i$ and $f_1 g_1 = g_2 f_2$.*

Proof. Following [Coh06, Section 2.7], continuant polynomials $\mathfrak{p}_n \in \Bbbk\langle y_1, \dots, y_n\rangle$ are recursively defined as

$$\mathfrak{p}_0 = 1, \qquad \mathfrak{p}_1 = y_1, \qquad \mathfrak{p}_n = \mathfrak{p}_{n-1} y_n + \mathfrak{p}_{n-2} \quad \text{for} \quad n \geq 2.$$

By [Coh06, Proposition 2.7.6] there exist $\alpha_1, \alpha_2 \in \Bbbk \setminus \{0\}$, $r \in \mathbb{N}$ and $a_1, \dots, a_r \in \Bbbk\langle\underline{x}\rangle$ such that

$$f_1 = \alpha_1 \mathfrak{p}_r(a_1, \dots, a_r), \qquad f_2 = \alpha_2 \mathfrak{p}_r(a_r, \dots, a_1)$$

and a_i are nonconstant for $1 < i < r$. If $a_r = 0$, then

$$\mathfrak{p}_r(a_1, \dots, a_r) = \mathfrak{p}_{r-2}(a_1, \dots, a_{r-2}), \quad \mathfrak{p}_r(a_r, \dots, a_1) = \mathfrak{p}_{r-2}(a_{r-2}, \dots, a_1).$$

If $a_r \in \Bbbk \setminus \{0\}$, then an easy manipulation of the recursive relation for $\mathfrak{p}_n$ yields

$$\mathfrak{p}_r(a_1, \dots, a_r) = a_r \mathfrak{p}_{r-1}(a_1, \dots, a_{r-1} + \tfrac{1}{a_r}), \quad \mathfrak{p}_r(a_r, \dots, a_1) = a_r \mathfrak{p}_{r-1}(a_{r-1} + \tfrac{1}{a_r}, \dots, a_1).$$

Analogous conclusions hold for $a_1 \in \Bbbk$. Hence there exist $\beta_1, \beta_2 \in \Bbbk$ and nonconstant $b_1, \dots, b_s \in \Bbbk\langle\underline{x}\rangle$ such that

$$f_1 = \beta_1 \mathfrak{p}_s(b_1, \dots, b_s), \qquad f_2 = \beta_2 \mathfrak{p}_s(b_s, \dots, b_1).$$

Since

$$\mathfrak{p}_s(b_1, \dots, b_s)\mathfrak{p}_{s-1}(b_{s-1}, \dots, b_1) = \mathfrak{p}_{s-1}(b_1, \dots, b_{s-1})\mathfrak{p}_s(b_s, \dots, b_1)$$

holds by [Coh06, Lemma 2.7.2] and the degree of a continuant polynomial in nonconstant arguments equals the sum of degrees of its arguments by the recursive relation,

$$b_1 = \tfrac{1}{\beta_1}\mathfrak{p}_{s-1}(b_{s-1}, \dots, b_1), \quad b_2 = \tfrac{1}{\beta_2}\mathfrak{p}_{s-1}(b_1, \dots, b_{s-1})$$

satisfy $\deg b_1 = \deg b_2 < \deg f_1 = \deg f_2$. □

Remark 2.3. While probably known to the specialists for factorization in free algebras, Lemma 2.2 implies that checking whether two irreducible polynomials are stably associated corresponds to solving a (finite) linear system.

Let $\Omega^n = (\Omega_1^n, \dots, \Omega_d^n)$ be a tuple of **generic** $n \times n$ **matrices** whose dn^2 entries are commuting independent variables are viewed as coordinates of the affine space $\mathrm{M}_n(\mathbb{k})^d$.

Lemma 2.4 ([HKV, Lemma 2.2]). *If* $f \in \mathbb{k}\langle\underline{x}\rangle$ *is nonconstant, then* $\det f(\Omega^{(n)})$ *is nonconstant for large enough* $n \in \mathbb{N}$.

3. Free Bertini's theorem

In this section we prove our first main result (Theorem 3.2). First we show that a certain linear equation in a free algebra has a unique solution (up to a scalar multiple).

Lemma 3.1. *Let* $f, g \in \mathbb{k}\langle\underline{x}\rangle$ *and assume* f *is not composite. If nonzero* $\alpha \in \mathbb{k}$ *and* $b_1, b_2 \in \mathbb{k}\langle\underline{x}\rangle$ *satisfy*

$$fb_1 = b_1 g, \quad fb_2 = \alpha b_2 g, \quad \deg b_1 = \deg b_2 < \deg f,$$

then $\alpha = 1$ *and* $b_2 \in \mathbb{k} b_1$.

Proof. Since f is not composite, its centralizer in $\mathbb{k}\langle\underline{x}\rangle$ equals $\mathbb{k}[f]$ by [Ber69, Theorem 5.3]. Therefore its centralizer in $\mathbb{k}(\!\langle\underline{x}\rangle\!)$, the universal skew field of fractions of $\mathbb{k}\langle\underline{x}\rangle$ (see [Coh06, Chapter 7] for more information), equals $\mathbb{k}(f)$ by [Coh06, Theorem 7.9.8 and Proposition 3.2.9]. Since

$$b_1^{-1} f b_1 = g = \alpha^{-1} b_2^{-1} f b_2,$$

we have $\det f(\Omega^{(n)}) = \alpha^{-n} \det f(\Omega^{(n)})$ for large enough n by Lemma 2.4, so $\alpha = 1$ and $b_2 b_1^{-1} \in \mathbb{k}(\!\langle\underline{x}\rangle\!)$ commutes with f. Hence there exist univariate coprime polynomials $q_1, q_2 \in \mathbb{k}[t]$ such that $b_2 b_1^{-1} = q_1(f)^{-1} q_2(f)$, and consequently $q_1(f) b_2 = q_2(f) b_1$. Let $b \in \mathbb{k}\langle\underline{x}\rangle$ be such that $b_i = c_i b$ for right coprime $c_1, c_2 \in \mathbb{k}\langle\underline{x}\rangle$. Then

$$q_1(f) c_2 = q_2(f) c_1,$$

so $q_1(f)$ and c_1 are stably associated. Therefore $\deg q_1(f) = \deg c_1$ by Lemma 2.2. Since the degree of $q_1(f)$ is either 0 or at least $\deg f$, and $\deg c_1 \le \deg b_1 < \deg f$, we conclude $\deg c_1 = 0$. Hence c_1, c_2 are (nonzero) scalars. □

The proof of free Bertini's theorem is based on Bergman's centralizer theorem [Ber69]. While otherwise inherently different from ours, Stein's proof of (the special case of) the classical Bertini's theorem in two commuting variables [Ste89] also uses "centralizers" with respect to the Poisson bracket on $\mathbb{k}[t_1, t_2]$.

Theorem 3.2. *Let* $\overline{\mathbb{k}}$ *be the algebraic closure of a field* $\mathbb{k}$. *The following are equivalent for* $f \in \mathbb{k}\langle\underline{x}\rangle \setminus \mathbb{k}$:

(i) $f - \lambda$ *factors in* $\overline{\mathbb{k}}\langle\underline{x}\rangle$ *for infinitely many* $\lambda \in \overline{\mathbb{k}}$;
(ii) $f - \lambda$ *factors in* $\overline{\mathbb{k}}\langle\underline{x}\rangle$ *for all* $\lambda \in \overline{\mathbb{k}}$;
(iii) the centralizer of f *in* $\mathbb{k}\langle\underline{x}\rangle$ *is strictly larger than* $\mathbb{k}[f]$;
(iv) f *is composite over* $\mathbb{k}$.

Proof. Implications (iv)⇒(ii)⇒(i) are clear, and (iii)⇒(iv) is a restatement of [Ber69, Theorem 5.3]. Thus it suffices to prove (i)⇒(iii).

Let $\Lambda \subseteq \overline{\mathbb{k}}$ be an infinite set of λ such that $f-\lambda$ factors in $\overline{\mathbb{k}}\!<\!\underline{x}\!>$. For each such λ there exist nonconstant $p_\lambda, q_\lambda \in \overline{\mathbb{k}}\!<\!\underline{x}\!>$ such that $f-\lambda = p_\lambda q_\lambda$. Observe that

$$f p_\lambda = (\lambda + p_\lambda q_\lambda) p_\lambda = p_\lambda (q_\lambda p_\lambda + \lambda) \tag{3.1}$$

for all $\lambda \in \Lambda$. Since $\deg p_\lambda < \deg f$ for all $\lambda \in \Lambda$, there exists an infinite subset $\Lambda_0 \subseteq \Lambda \setminus \{0\}$ and $\delta < \deg f$ such that $\deg p_\lambda = \delta$ for all $\lambda \in \Lambda_0$. Furthermore, $\lambda + p_\lambda q_\lambda, p_\lambda$ are left coprime and $p_\lambda, q_\lambda p_\lambda + \lambda$ are right coprime whenever $\lambda \neq 0$. Therefore f and $q_\lambda p_\lambda + \lambda$ are stably associated for every $\lambda \in \Lambda_0$. By Proposition 2.1, there are (up to a scalar multiple) only finitely many polynomials stably associated to f. Hence there exist distinct $\mu, \nu \in \Lambda_0$ such that $q_\nu p_\nu + \nu$ is a scalar multiple of $q_\mu p_\mu + \mu$. Suppose f is not composite over $\overline{\mathbb{k}}$. Then p_ν is a scalar multiple of p_μ by (3.1) and Lemma 3.1. However, this is impossible since

$$p_\mu q_\mu - p_\nu q_\nu = \nu - \mu \in \overline{\mathbb{k}} \setminus \{0\}.$$

Therefore f is composite over $\overline{\mathbb{k}}$. In particular,

$$U := \{p \in \overline{\mathbb{k}}\!<\!\underline{x}\!>\colon\ \deg p < \deg f,\ p(0)=0,\ fp - pf = 0\} \neq \{0\}.$$

But U is a subspace given by equations over $\mathbb{k}$, so $U \cap \mathbb{k}\!<\!\underline{x}\!> \neq \{0\}$. Now (iii) follows because $U \cap \mathbb{k}[f] = \{0\}$ and U is contained in the centralizer of f in $\mathbb{k}\!<\!\underline{x}\!>$. □

A slightly stronger version holds for homogeneous polynomials.

Corollary 3.3. *Let $f \in \mathbb{k}\!<\!\underline{x}\!> \setminus \mathbb{k}$ be homogeneous. Then $f-1$ factors in $\overline{\mathbb{k}}\!<\!\underline{x}\!>$ if and only if $f = f_0^n$ for some $n > 1$ and homogeneous $f_0 \in \mathbb{k}\!<\!\underline{x}\!>$.*

Proof. If $f-1$ factors in $\overline{\mathbb{k}}\!<\!\underline{x}\!>$, then $f - \lambda^{\deg f}$ factors in $\overline{\mathbb{k}}\!<\!\underline{x}\!>$ for every $\lambda \in \overline{\mathbb{k}}$ because it is up to a linear change of variables equal to $f(\lambda x) - \lambda^{\deg f} = \lambda^{\deg f}(f-1)$. Therefore f is composite by Theorem 3.2, and furthermore a power by homogeneity. □

4. Eigenlevel sets

Throughout this section let $\mathbb{k}$ be an algebraically closed field of characteristic 0. Recall the definition of the eigenlevel set of f at λ,

$$L_\lambda(f) = \bigcup_{n\in\mathbb{N}} \left\{X \in \mathrm{M}_n(\mathbb{k})^d \colon \lambda \text{ is an eigenvalue of } f(X)\right\}.$$

In the terminology of [HKV18, HKV], $L_\lambda(f)$ is the free locus of $f-\lambda$. Combined with existing irreducibility results for free loci of noncommutative polynomials [HKV18, HKV], free Bertini's theorem becomes a geometric statement about eigenlevel sets.

Corollary 4.1. *If $f \in \mathbb{k}\!<\!\underline{x}\!>\setminus\mathbb{k}$ is not composite, then there exists $N \in \mathbb{N}$ such that for all but finitely many $\lambda \in \mathbb{k}$,*

$$\left\{X \in \mathrm{M}_n(\mathbb{k})^d \colon \lambda \textit{ is an eigenvalue of } f(X)\right\} \tag{4.1}$$

is a reduced and irreducible hypersurface in $\mathrm{M}_n(\mathbb{k})^g$ for all $n \geq N$.

Proof. By Theorem 3.2, there is a cofinite subset Λ of $\Bbbk\setminus\{f(0)\}$ such that $f-\lambda$ is irreducible over $\Bbbk$ for $\lambda\in\Lambda$. By [HKV18, Theorem 4.3] for each $\lambda\in\Lambda$ there exists $N_\lambda\in\mathbb{N}$ such that the hypersurface (4.1) is reduced and irreducible for every $n\geq N_\lambda$. However, since polynomials $f-\lambda$ for $\lambda\in\Bbbk$ only differ in the constant part, it follows by [HKV18, Remark 3.5 and proof of Lemma 4.2] that one can choose $N=N_\lambda$ independent of λ. □

Remark 4.2. Let $p_1,p_2\in\Bbbk[t]$. Then $p_2\in\Bbbk[p_1]$ if and only if for every $\lambda_1\in\Bbbk$ there exists $\lambda_2\in\Bbbk$ such that every zero of $p_1-\lambda_1$ is a zero of $p_2-\lambda_2$. Indeed, $p_1-\lambda_1$ has only simple zeros for infinitely many λ_1, in which case $\{p_1-\lambda_1=0\}\subseteq\{p_2-\lambda_2=0\}$ implies that $p_1-\lambda_1$ divides $p_2-\lambda_2$. Then the claim follows from the division algorithm in $\Bbbk[t]$ by induction on $\deg p_2$.

Theorem 4.3. *For $f,g\in\Bbbk<\underline{x}>$ the following are equivalent:*

(i) each eigenlevel set of f is contained in an eigenlevel set of g;
(ii) there exist $p\in\Bbbk[t]$ and nonzero $a,h\in\Bbbk<\underline{x}>$ such that $g=p(h)$ and $fa=ah$.

Proof. (ii)⇒(i) By Lemma 2.4,

$$h(\Omega^n)=a(\Omega^n)^{-1}f(\Omega^n)a(\Omega^n)$$

for all large enough n, and thus

$$\det(h(\Omega^n)-\lambda I)=\det(f(\Omega^n)-\lambda I)$$

for all $\lambda\in\Bbbk$ and $n\in\mathbb{N}$. Hence $L_\lambda(f)=L_\lambda(h)$ for all $\lambda\in\Bbbk$. Since every univariate polynomial over $\Bbbk$ factors into linear factors, each eigenlevel set of f is contained in an eigenlevel set of $p(h)$.

(i)⇒(ii) Assume that f,g are nonconstant. Then $f=p_1(h_1)$ and $g=p_2(h_2)$ for some $p_1,p_2\in\Bbbk[t]$ and non-composite $h_1,h_2\in\Bbbk<\underline{x}>$ with $h_1(0)=0=h_2(0)$. By Corollary 4.1 there is a cofinite set $\Lambda\subseteq\Bbbk$ such that $L_\lambda(h_1)\cap \mathrm{M}_n(\Bbbk)^d$ and $L_\lambda(h_2)\cap \mathrm{M}_n(\Bbbk)^d$ are reduced and irreducible hypersurfaces for all $\lambda\in\Lambda$ and large enough $n\in\mathbb{N}$. Since eigenlevel sets of f are contained in eigenlevel sets of g, there are infinitely many pairs $(\lambda_1,\lambda_2)\in\Lambda^2$ such that $L_{\lambda_1}(h_1)=L_{\lambda_2}(h_2)$. By comparing

$$\det(h_1(\Omega^{(n)})-\lambda_1 I),\quad \det(h_2(\Omega^{(n)})-\lambda_2 I)$$

one can replace h_2 with $\alpha h_2+\beta$ for some $\alpha\in\Bbbk\setminus\{0\}$ and $\beta\in\Bbbk$ (and change p_2 accordingly) so that

$$\det(h_1(\Omega^{(n)})-\lambda I)=\det(h_2(\Omega^{(n)})-\lambda I) \tag{4.2}$$

for all $\lambda\in\Bbbk$ and $n\in\mathbb{N}$. By [HKV18, Theorem 4.3], $h_1-\lambda$ and $h_2-\lambda$ are stably associated for all $\lambda\in\Lambda$. Let $\delta=\deg h_1$. By Lemma 2.2 there exist nonzero $a_\lambda,b_\lambda\in\Bbbk<\underline{x}>$ of degree less than δ for $\lambda\in\Lambda$ such that

$$(h_1-\lambda)a_\lambda=b_\lambda(h_2-\lambda). \tag{4.3}$$

Since (4.3) is a linear system in (a_λ,b_λ) with a rational parameter λ, there exist nonzero $A,B\in\Bbbk[t]\otimes\Bbbk<\underline{x}>$ of degree (with respect to x) less than δ such that

$$(h_1-t)A=B(h_2-t).$$

By looking at the degree of A with respect to t one can find $C \in \mathbb{k}[t] \otimes \mathbb{k}\langle\underline{x}\rangle$ such that $a := A - C(t - h_2) \in \mathbb{k}\langle\underline{x}\rangle$. Note that $a \neq 0$ since $\deg A < \delta = \deg h_2$. Letting $b := B - (t - h_1)C$ we obtain

$$(h_1 - t)a = b(h_2 - t). \tag{4.4}$$

By comparing degrees with respect to t in (4.4) we get $b \in \mathbb{k}\langle\underline{x}\rangle$ and consequently $a = b$. For $h := p_1(h_2)$ we thus have

$$fa = p_1(h_1)a = ap_1(h_2) = ah.$$

Finally, since for every $\lambda_1 \in \mathbb{k}$ there exists $\lambda_2 \in \mathbb{k}$ such that

$$L_{\lambda_1}(p_1(h_2)) = L_{\lambda_1}(h) = L_{\lambda_1}(f) \subseteq L_{\lambda_2}(g) = L_{\lambda_2}(p_2(h_2))$$

and $\det h_2(\Omega^{(n)})$ is nonconstant for large n by Lemma 2.4, Remark 4.2 implies $p_2 = p \circ p_1$ for some $p \in \mathbb{k}[t]$. □

Corollary 4.4. *Let $f, g \in \mathbb{k}\langle\underline{x}\rangle$. Then eigenlevel sets of f and g coincide if and only if there is a nonzero $a \in \mathbb{k}\langle\underline{x}\rangle$ such that $fa = ag$.*

Proof. If eigenlevel sets of f and g coincide, then $f = p_1(h_1)$, $g = p_2(h_2)$ and $h_1 a = ah_2$ for $0 \neq a, h_1, h_2 \in \mathbb{k}\langle\underline{x}\rangle$ as in the proof of Theorem 4.3. Furthermore,

$$L_\lambda(p_1(h_1)) = L_\lambda(p_2(h_2)) = L_\lambda(p_2(h_1))$$

implies $p_1 = p_2$ and therefore $fa = ag$. For the converse see the proof of (ii)⇒(i) in Theorem 4.3. □

Example 4.5. Let

$$f = x_1 + x_2 + x_1x_2^2, \quad g = x_1 + x_2 + x_2^2x_1, \quad a = 1 + x_1^2 + x_1x_2 + x_2x_1 + x_1x_2^2x_1.$$

Then $fa = ag$, so eigenlevel sets of f and g coincide. Note that $\deg a > \deg f$. While

$$f(1 + x_2x_1) = (1 + x_1x_2)g$$

holds, which complies with Lemma 2.2, there is no $b \in \mathbb{k}\langle\underline{x}\rangle$ such that $fb = bg$ and $\deg b \leq \deg f$.

5. Locally quasiconvex polynomials

On the free $\mathbb{R}$-algebra $\mathbb{R}\langle\underline{x}\rangle$ there is a unique involution $*$ satisfying $x_j^* = x_j$. A noncommutative polynomial $f \in \mathbb{R}\langle\underline{x}\rangle$ is **symmetric** if $f^* = f$. Let $\mathbb{S}^d = \bigcup_{n\in\mathbb{N}} \mathrm{S}_n(\mathbb{R})^d$. Then f is symmetric if and only if $f(X) \in \mathbb{S}^1$ for all $X \in \mathbb{S}^d$. By $A \succ 0$ (resp. $A \succeq 0$) we denote that $A \in \mathbb{S}^1$ is positive definite (resp. semidefinite).

Let $f \in \mathbb{R}\langle\underline{x}\rangle$ be symmetric. As in [HM12] (cf. [HKMV]) we define its **positivity domain**,

$$\mathcal{D}(f) = \bigcup_{n\in\mathbb{N}} \mathcal{D}_n(f)$$

where $\mathcal{D}_n(f)$ is the closure of the connected component of

$$\{X \in \mathrm{S}_n(\mathbb{R})^d : f(X) \succ 0\}$$

containing the origin $0^d \in \mathrm{S}_n(\mathbb{R})^d$. It is known [HM12] that $\mathcal{D}(f)$ is convex (i.e., $\mathcal{D}_n(f)$ is convex for all $n \in \mathbb{N}$) if and only if $\mathcal{D}(f)$ is the solution set of a linear matrix inequality. We will require the following version of [HKMV, Theorem 1.5].

Proposition 5.1. *Let $f \in \mathbb{R}{<}\underline{x}{>}$ be symmetric and irreducible over $\mathbb{C}$, with $f(0) = 0$. If $\mathcal{D}(1-f)$ is proper and convex, then*

$$f = \ell_0 + \ell_1^2 + \cdots + \ell_m^2 \tag{5.1}$$

for some linear $\ell_0, \dots, \ell_m \in \mathbb{R}{<}\underline{x}{>}$.

Proof. Let $\underline{y} = (y_1, \dots, y_d)$ and $\underline{y}^* = (y_1^*, \dots, y_d^*)$ be freely noncommuting variables, and consider $\mathbb{C}{<}\underline{y}, \underline{y}^*{>}$ with the $\mathbb{R}$-linear involution $*$ sending y_j to y_j^* and acting on $\mathbb{C}$ as the complex conjugate. Since $f \in \mathbb{R}{<}\underline{x}{>}$ is symmetric and irreducible over $\mathbb{C}$, the noncommutative polynomial $\tilde{f} := f(y_1 + y_1^*, \dots, y_d + y_d^*) \in \mathbb{C}{<}\underline{y}, \underline{y}^*{>}$ is hermitian and irreducible in $\mathbb{C}{<}\underline{y}, \underline{y}^*{>}$. The positivity domain of $1 - \tilde{f}$ (see [HKMV]) is the union over $n \in \mathbb{N}$ of closures of connected components of

$$\{(Y, Y^*) \in \mathrm{M}_n(\mathbb{C})^d \times \mathrm{M}_n(\mathbb{C})^d \colon I - f(Y_1 + Y_1^*, \dots, Y_d + Y_d^*) \succ 0\}$$

containing the origin. Furthermore, as $\mathcal{D}(1-f)$ is proper and convex, the standard embedding of hermitian $n \times n$ matrices into symmetric $(2n) \times (2n)$ matrices implies that $\mathcal{D}(1 - \tilde{f})$ is also proper and convex. Therefore

$$\tilde{f} = \tilde{\ell}_0 + \sum_{k>0} \tilde{\ell}_k^* \tilde{\ell}_k$$

for some linear $\tilde{\ell}_k \in \mathbb{C}{<}\underline{y}, \underline{y}^*{>}$ by [HKMV, Theorem 1.5]. Note that $f = \tilde{f}(x/2, x/2)$. Since $\tilde{f}$ is hermitian, $\tilde{\ell}$ is hermitian, so $\tilde{\ell}_0(x/2, x/2)$ is symmetric. Furthermore,

$$\tilde{\ell}_k^* \tilde{\ell}_k = (\operatorname{re} \tilde{\ell}_k)^2 + (\operatorname{im} \tilde{\ell}_k)^2 + i[\operatorname{re} \tilde{\ell}_k, \operatorname{im} \tilde{\ell}_k]$$

for $k > 0$; since f is symmetric, $\sum_{k>0} \tilde{\ell}_k(x/2, x/2)^* \tilde{\ell}_k(x/2, x/2)$ is a sum of squares in $\mathbb{R}{<}\underline{x}{>}$. Hence f is of the form (5.1). □

Remark 5.2. If f is of the form (5.1), then it is easy to present $\mathcal{D}(1-f)$ as the solution set of a linear matrix inequality, so $\mathcal{D}(1-f)$ is convex.

Lemma 5.3. *Let $h = \ell_0 + \sum_{k>0} \ell_k^2$ for some linear $\ell_k \in \mathbb{R}{<}\underline{x}{>}$, and let $\underline{t} = (t_1, \dots, t_d)$ be the coordinates of $\mathbb{R}^d$.*

(i) If $\beta > 0$, then $h + \beta$ is a sum of squares in $\mathbb{R}{<}\underline{x}{>}$ if and only if $h(\underline{t}) + \beta$ is a sum of squares in $\mathbb{R}[\underline{t}]$.
(ii) If $\mathcal{D}_1(\alpha - h) \subseteq \mathcal{D}_1(\beta + h)$ for some $\alpha, \beta > 0$, then $\beta + h$ is a sum of squares.

Proof. (i) Observe that $h + \beta$ has a unique representation $h + \beta = v^* S v$, where $S \in \mathrm{S}_{d+1}(\mathbb{R})$ and $v^* = (1, x_1, \dots, x_d)$. It is easy to see that $h(\underline{t}) + \beta$ is a sum of squares in $\mathbb{R}[t]$ if and only if $S \succeq 0$, which is further equivalent to $h + \beta$ being a sum of squares in $\mathbb{R}{<}\underline{x}{>}$.

(ii) Since $\mathcal{D}_1(\alpha - h)$ is convex, we have

$$h(\tau) \le \alpha \;\Rightarrow\; h(\tau) \ge -\beta$$

for all $\tau \in \mathbb{R}^d$. That is, an upper bound on $h(\underline{t})$ implies a lower bound on $h(\underline{t})$, which is clearly possible only if $h(\tau) \geq -\beta$ for all $\tau \in \mathbb{R}^d$. Since $h(\underline{t})+\beta$ is a quadratic nonnegative polynomial, it is a sum of squares in $\mathbb{R}[\underline{t}]$. Now (ii) follows by (i). $\square$

Recall that a symmetric $f \in \mathbb{R}\langle\underline{x}\rangle$ with $f(0) = 0$ is **locally quasiconvex** if there exists $\varepsilon > 0$ such that $\mathcal{D}(\lambda - f)$ is convex for every $\lambda \in (0, \varepsilon)$.

Theorem 5.4. *Le $f \in \mathbb{R}\langle\underline{x}\rangle$ be symmetric with $f(0) = 0$. The following are equivalent:*

(i) f is locally quasiconvex;
(ii) $\mathcal{D}(\lambda - f)$ is convex for every $\lambda > 0$;
(iii) $-f$ is a sum of hermitian squares; or

$$f = p(\ell_0 + \ell_1^2 + \cdots + \ell_m^2) \tag{5.2}$$

for $p \in \mathbb{R}[t]$ with $p(0) = 0$ and linear $\ell_0, \dots, \ell_m \in \mathbb{R}\langle\underline{x}\rangle$ satisfying one of the following:
(a) $p(\tau) \leq 0$ for $\inf_{\mathbb{R}^d}(\ell_0 + \ell_1^2 + \cdots + \ell_m^2) < \tau < 0$,
(b) $\ell_k = 0$ for all $k > 0$.

Proof. (ii)⇒(i) Clear.

(i)⇒(iii) Let $\varepsilon > 0$ be such that $\mathcal{D}(\lambda - f)$ is convex for every $\lambda \in (0, \varepsilon)$. If $\mathcal{D}(\lambda - f) = \mathbb{S}^d$ for all such λ, then $-f(X)$ is positive semidefinite for every $X \in \mathbb{S}^d$, so $-f$ is a sum of hermitian squares by [Hel02, McC01]. Otherwise we can without loss of generality assume that $\mathcal{D}(\lambda - f) \neq \mathbb{S}^d$ for $\lambda \in (0, \varepsilon)$. If $\lambda - f$ is irreducible over $\mathbb{C}$ for some such λ, then f is of the form (5.1) by Proposition 5.1, and (a) holds with $p = t$. If $\lambda - f$ factors in $\mathbb{R}\langle\underline{x}\rangle$ for all $\lambda \in (0, \varepsilon)$, then $f = p(h)$ for some $p \in \mathbb{R}[t]$ and a non-composite $h \in \mathbb{R}\langle\underline{x}\rangle$ with $p(0) = 0 = h(0)$ by Theorem 3.2. Since f is symmetric, h is also symmetric because it is unique up to a scalar multiple. Furthermore, $-p$ is not a sum of squares since $-f$ is not a sum of hermitian squares.

Let us introduce some auxiliary notation. If $\lambda - p$ attains a negative value on $(0, \infty)$, let $\pi_\lambda \geq 0$ be such that

$$(\lambda - p)|_{[0,\pi_\lambda]} \geq 0, \qquad \exists \varepsilon' > 0\colon (\lambda - p)|_{(\pi_\lambda, \pi_\lambda + \varepsilon')} < 0.$$

If $\lambda - p$ attains a negative value on $(-\infty, 0)$, let $\nu_\lambda \leq 0$ be such that

$$(\lambda - p)|_{[\nu_\lambda,0]} \geq 0, \qquad \exists \varepsilon' > 0\colon (\lambda - p)|_{(\nu_\lambda - \varepsilon', \nu_\lambda)} < 0.$$

Then π_λ, ν_λ are zeros of $\lambda - p$ and strictly monotone functions in λ, continuous for λ close to 0.

We distinguish two cases. First suppose that $-p$ is nonnegative on $(-\infty, 0)$ or $(0, \infty)$. By replacing $p(t), h$ with $p(-t), -h$ if necessary, we can assume that $-p$ attains a negative value on $(0, \infty)$. Then

$$\mathcal{D}(\lambda - f) = \mathcal{D}(\pi_\lambda - h)$$

for all small enough $\lambda > 0$. Since h is not composite, $\pi_\lambda - h$ is irreducible for all but finitely many λ by Theorem 3.2. Because $\mathcal{D}(\lambda - f)$ is convex, h is of the form (5.1) by Proposition 5.1, so (a) holds.

Now suppose that $-p$ attains negative values on$(-\infty, 0)$ and $(0, \infty)$. Then

$$\mathcal{D}(\lambda - f) = \mathcal{D}(-\nu_\lambda + h) \cap \mathcal{D}(\pi_\lambda - h) \tag{5.3}$$

for all small enough $\lambda > 0$. Suppose that one the sets $\mathcal{D}(-\nu_\lambda + h)$ and $\mathcal{D}(\pi_\lambda - h)$ is contained in the other. By replacing $p(t), h$ with $p(-t), -h$ if necessary, we can assume that $\mathcal{D}(\pi_\lambda - h) \subseteq \mathcal{D}(-\nu_\lambda + h)$. Since h is not composite, $\pi_\lambda - h$ is irreducible for all but finitely many λ, so h is of the form (5.1) by convexity of $\mathcal{D}(\lambda - f)$ and Proposition 5.1. By Lemma 5.3, $-\nu_\lambda + h$ is a sum of squares. If $\mu = -\lim_{\lambda\downarrow 0} \nu_\lambda$, then $\mu + h$ is nonnegative on $\mathbb{R}^d$ and $-p$ is nonnegative on $[-\mu, 0]$, so (a) holds. Finally we are left with the scenario where the intersection (5.3) is irredundant. Then $\mathcal{D}(-\nu_\lambda + h)$ and $\mathcal{D}(\pi_\lambda - h)$ are both convex by [HKMV, Corollary 1.2]. By Proposition 5.1 we conclude that h is linear, so (b) holds.

(iii)⇒(ii) If $-f$ is a sum of hermitian squares, then $\mathcal{D}(\lambda - f) = \mathbb{S}^d$ for every $\lambda > 0$. Otherwise let f be as in (5.2). Then $\mathcal{D}(\lambda - f)$ equals one of

$$\mathbb{S}^d, \quad \mathcal{D}(\pi_\lambda - h), \quad \mathcal{D}(-\nu_\lambda + h), \quad \mathcal{D}(-\nu_\lambda + h) \cap \mathcal{D}(\pi_\lambda - h),$$

depending on the existence of ν_λ, π_λ. Note that $\mathcal{D}(\pi_\lambda - h)$ is always convex by Remark 5.2. If (b) holds, then $\mathcal{D}(\lambda - f)$ is convex for $\lambda > 0$ since h is linear and intersection of convex sets is again convex. If (a) holds, then $\nu_\lambda \le \inf_{\mathbb{R}^d}(\ell_0 + \sum_{k>0} \ell_k^2)$, so $\mathcal{D}(-\nu_\lambda + h) = \mathbb{S}^d$ and $\mathcal{D}(\lambda - f)$ is convex for $\lambda > 0$. □

Remark 5.5. Few comments on the condition (a) in Theorem 5.4 are in order. Let

$$\mu = \inf_{\mathbb{R}^d}(\ell_0 + \ell_1^2 + \cdots + \ell_m^2).$$

Then $\mu > -\infty$ if and only if ℓ_0 lies in the linear span of $\ell_1, \dots, \ell_m$; more precisely, if $\ell_1, \dots, \ell_m$ are linearly independent and $\ell_0 = \alpha_1\ell_1 + \cdots + \alpha_m\ell_m$, then $-4\mu = \alpha_1^2 + \cdots + \alpha_m^2$. This follows from considering $\ell_0 + \ell_1^2 + \cdots + \ell_m^2 + \mu = v^*Sv$ for $v^* = (1, x_1, \dots, x_d)$ and $S \succeq 0$ as in the proof of Lemma 5.3(i). Furthermore, using an algebraic certificate for nonnegativity [Mar08, Prop 2.7.3], the condition (a) can also be stated as follows. Let $S \subset \mathbb{R}[t]$ be the convex cone of sums of (two) squares. If $\mu = -\infty$, then

$$\sup_{(-\infty,0]} p = p(0) = 0 \quad \iff \quad p \in t(S - tS);$$

and if $-\infty < \mu \le 0$, then

$$\max_{[\mu,0]} p = p(0) = 0 \quad \iff \quad p \in t(S - tS + (t-\mu)(S - tS)).$$

Remark 5.6. Another aspect of Theorem 5.4 is the following. Proposition 5.1 states that every irreducible symmetric polynomial with a convex positivity domain is quadratic (and concave). On the other hand, there is no shortage of reducible symmetric polynomials that contain a factor of degree at least 3 and have convex positivity domain; see [HKMV, Sections 5 and 6]. However, if the constant term of such a polynomial is slightly perturbed, then its positivity domain is no longer convex by Theorem 5.4.

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