Real free loci of linear matrix pencils

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(joint work with Igor Klep)

Let $A_1,\dots,A_g \in \mathrm{M}_d(\mathbb{C})$. The formal affine linear combination $L = I - \sum_j A_j x_j$, where x_j are freely noncommuting variables, is called a **(monic) linear pencil** of size d. If all A_j are hermitian matrices, then L is a **hermitian** pencil. Linear pencils appear in various areas, from matrix theory and real algebraic geometry to convex optimization and control theory. In the spirit of free real algebraic geometry and free analysis, the evaluation of L at a point $X = (X_1,\dots,X_g) \in \mathrm{M}_n(\mathbb{C})^g$ is defined using the (Kronecker) tensor product

$$L(X) = I \otimes I - \sum_{i=1}^{g} A_i \otimes X_i \in \mathrm{M}_{nd}(\mathbb{C}),$$

giving rise to the **free (singular) locus**,

$$\mathscr{Z}(L) = \bigcup_{n\in\mathbb{N}} \mathscr{Z}_n(L), \qquad \text{where} \quad \mathscr{Z}_n(L) = \{X \in \mathrm{M}_n(\mathbb{C})^g \colon \det(L(X)) = 0\}.$$

Clearly, each $\mathscr{Z}_n(L)$ is an algebraic subset of $\mathrm{M}_n(\mathbb{C})^g$. If L is a hermitian pencil, $\mathscr{Z}_n(L)$ is closed under conjugate transposition and thus has a natural real structure. In this case we also consider the real points of $\mathscr{Z}_n(L)$, namely the set of tuples of hermitian matrices in $\mathscr{Z}_n(L)$, denoted $\mathscr{Z}_n^{\mathrm{h}}(L)$. The set

$$\mathscr{Z}^{\mathrm{h}}(L) = \bigcup_{n\in\mathbb{N}} \mathscr{Z}_n^{\mathrm{h}}(L)$$

is called the **free real locus** of L. Zariski closures of boundaries of free spectrahedra [2], singularity sets of noncommutative rational functions [3, 4], and "free real algebraic hypersurfaces" are examples of free real loci.

In [6, Theorems 3.6 and 5.4] we solved the inclusion problem for free real loci in terms of algebras generated by the coefficients of the corresponding pencils. For $L = I - \sum_i A_i x_i$ and $L' = I - \sum_i A'_i x_i$ of sizes d and d', respectively, let $\mathcal{A} \subseteq \mathrm{M}_d(\mathbb{C})$ and $\mathcal{A}' \subseteq \mathrm{M}_{d'}(\mathbb{C})$ be the $\mathbb{C}$-algebras generated by $A_1,\dots,A_g$ and $A'_1,\dots,A'_g$, respectively. Let $\operatorname{rad}\mathcal{A}$ denote the Jacobson radical of $\mathcal{A}$.

Theorem 1. *Let L and $\widetilde{L}$ be as above. Then $\mathscr{Z}(L) \subseteq \mathscr{Z}(\widetilde{L})$ if and only if there exists a homomorphism $\widetilde{\mathcal{A}}/\operatorname{rad}\widetilde{\mathcal{A}} \to \mathcal{A}/\operatorname{rad}\mathcal{A}$ induced by $\widetilde{A}_i \mapsto A_i$.*

If L and $\widetilde{L}$ are hermitian, then $\mathscr{Z}^{\mathrm{h}}(L) \subseteq \mathscr{Z}^{\mathrm{h}}(\widetilde{L})$ if and only if there exists a $*$-homomorphism $\widetilde{\mathcal{A}} \to \mathcal{A}$ induced by $\widetilde{A}_i \to A_i$.

The proof proceeds in three steps. First we show that the Jacobson radical of the coefficient algebra is irrelevant for the free locus using the theory of trace identities on $n \times n$ matrices. Then we use an algebraization trick to relate the multiplicative structure of the coefficient algebra with points in the free locus. Finally, the second part of Theorem 1 follows by the properties of hyperbolic polynomials, i.e., the real variety $\mathscr{Z}_n(L)$ is Zariski dense in $\mathscr{Z}_n(L)$. In particular, we see that for a hermitian pencils, the inclusion of free loci $\mathscr{Z}(L) \subseteq \mathscr{Z}(\widetilde{L})$ is equivalent to the inclusion of free real loci $\mathscr{Z}^{\mathrm{h}}(L) \subseteq \mathscr{Z}^{\mathrm{h}}(\widetilde{L})$.

Theorem 1 presents the foundation for a more precise analysis of free loci. If the coefficients of L generate $\mathrm{M}_d(\mathbb{C})$, then we say that L is an **irreducible pencil**. If L and L' are irreducible pencils and $\mathscr{Z}(L) \subseteq \mathscr{Z}(L')$, then $\mathscr{Z}(L) = \mathscr{Z}(L')$. In this case L' and L are similar and moreover unitarily similar if they are hermitian [6, Theorem 3.11 and Corollary 5.5]. A free locus is **irreducible** if it is not a union of smaller free loci. By [6, Proposition 3.12], a free locus is irreducible if and only if it is a free locus of some irreducible pencil.

By applying Burnside's theorem on existence of subspaces to the coefficient algebra of the pencil it follows that every monic pencil L is similar to a pencil of the form

$$\begin{pmatrix} L_1 & \star & \cdots & \star \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & L_m & \star \\ 0 & \cdots & 0 & I \end{pmatrix}, \tag{1}$$

where L_k are irreducible pencils.

From the definition it does not follow that an irreducible free locus restricts to an irreducible hypersurface in $\mathrm{M}_n^g(\mathbb{C})$. Indeed, let

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then one can check that A_1, A_2 generate the full algebra of 3×3 matrices and

$$\det(I - \xi_1 A_1 - \xi_2 A_2) = (1 - \xi_1 + \xi_2)(1 - \xi_1 - \xi_2).$$

Hence $L = I - A_1 x_1 - A_2 x_2$ is an irreducible pencil but the surface $\mathscr{Z}_1(L)$ is not irreducible. However, in a forthcoming paper [5] we show that if $\mathscr{Z}(L)$ is an irreducible locus, then $\mathscr{Z}_n(L)$ is irreducible in $\mathrm{M}_n^g(\mathbb{C})$ for large enough n. Moreover, we prove that the determinant of the pencil is irreducible in the following sense. For $n \in \mathbb{N}$ let $\Xi^{(n)} = (\Xi_1^{(n)}, \dots, \Xi_g^{(n)})$ be the tuple of $n \times n$ generic matrices, i.e., matrices whose entries are independent commuting variables.

Theorem 2. *If L is an irreducible pencil, then there exists $n_0 \in \mathbb{N}$ such that $\det L(\Xi^{(n)})$ is an irreducible polynomial for all $n \geq n_0$.*

Let us say a few words about the proof. We apply the first fundamental theorem for the action of $\mathrm{GL}_n(\mathbb{C})$ on $\mathrm{M}_n^g(\mathbb{C})$ with simultaneous conjugation and the algebraization trick to establish the following: if $\det L'(\Xi^{(n)})$ is irreducible for all $n \geq n_1$, where $L' = I - \sum_j A_j x_j - A_{j'} A_{j''} x_{g+1}$, then $\det L(\Xi^{(n)})$ is irreducible for all $n \geq 2n_1$. Theorem 2 is then proved by induction on generation of $d \times d$ matrices by the coefficients of L and using the fact that the determinant of a generic matrix is irreducible. While the bound on n_0 constructed through the proof is exponential with respect to the size of L, one might believe that there exists a linear bound on n_0.

As an application of Theorem 2 we can inspect smooth points on the boundary of a free spectrahedron (LMI domain). Let L be a hermitian monic pencil of size d and let $\mathcal{D}_n(L)$ be the set of tuples of $n \times n$ hermitian matrices X making $L(X)$ positive semidefinite. The set

$$\mathcal{D}(L) = \bigcup_{n\in\mathbb{N}} \mathcal{D}_n(L)$$

is the **free spectrahedron** of L. Also denote

$$\partial\mathcal{D}(L) = \bigcup_{n\in\mathbb{N}} \partial\mathcal{D}_n(L), \qquad \partial\mathcal{D}_n(L) = \mathcal{D}_n(L) \cap \mathscr{Z}_n^{\mathrm{h}}(L),$$
$$\partial^1\mathcal{D}(L) = \bigcup_{n\in\mathbb{N}} \partial^1\mathcal{D}_n(L), \qquad \partial^1\mathcal{D}_n(L) = \{X \in \partial\mathcal{D}_n(L)\colon \dim\ker L(X) = 1\}.$$

Hence $\partial\mathcal{D}(L)$ is the boundary of the free spectrahedron $\mathcal{D}(L)$ and it is easy to see that if $\partial^1\mathcal{D}_n(L) \neq \emptyset$, then $\partial^1\mathcal{D}_n(L)$ are precisely the smooth points of $\partial\mathcal{D}_n(L)$. However, it is not a priori clear that $\partial^1\mathcal{D}(L)$ is nonempty and this question is related to a matrix theory problem known as Kippenhahn's conjecture.

A hermitian monic pencil L is **LMI-minimal** if it is of minimal size among all hermitian pencils L' satisfying $\mathcal{D}(L') = \mathcal{D}(L)$. Note that if L and L' are irreducible hermitian pencils, then $\mathscr{Z}(L) = \mathscr{Z}(L')$ implies $\mathcal{D}(L) = \mathcal{D}(L')$. Using Burnside's theorem and the hermitian structure of L we see that L is unitarily similar to $L_1 \oplus \cdots \oplus L_m$, where L_i are pairwise non-similar irreducible hermitian pencils.

For LMI-minimal hermitian pencils we can prove the following density result on smooth points.

Corollary 3. *Let L be a LMI-minimal hermitian pencil. Then there exists $n_0 \in \mathbb{N}$ such that $\partial^1\mathcal{D}_n(L)$ is Zariski dense in $\mathscr{Z}_n(L)$ for all $n \geq n_0$.*

The proof of Corollary 3 applies Theorem 2 and properties of hyperbolic polynomials. The density of $\partial^1\mathcal{D}(L)$ in $\mathcal{D}(L)$ is important in the study of free analytic maps between free spectrahedra [1].

References

[1] M. Augat, J. W. Helton, I. Klep, S. McCullough: *Bianalytic maps between free spectrahedra*, preprint `arXiv:1604.04952`.

[2] J. W. Helton, I. Klep, S. McCullough: *The matricial relaxation of a linear matrix inequality*, Math. Program. **138** (2013) 401–445.

[3] J. W. Helton, S. McCullough, V. Vinnikov: *Noncommutative convexity arises from linear matrix inequalities*, J. Funct. Anal. **240** (2006) 105–191.

[4] D. S. Kalyuzhnyi-Verbovetskiĭ, V. Vinnikov: *Singularities of rational functions and minimal factorizations: the noncommutative and the commutative setting*, Linear Algebra Appl. **430** (2009) 869–889.

[5] I. Klep, Š. Špenko, J. Volčič: *Invariant theory and geometry for free loci of matrix pencils*, in preparation.

[6] I. Klep, J. Volčič: *Free loci of matrix pencils and domains of noncommutative rational functions*, Comment. Math. Helv. **92** (2017) 105–130.