Free real algebraic geometry, with a focus on convexity

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One of the noncommutative analogs of the classical real algebraic geometry is free real algebraic geometry (FRAG), which studies noncommutative polynomials and rational functions, their evaluations on tuples of matrices, and positive definiteness thereof. The adjective "free" signals that one is interested in variables that are relation-free and matrix arguments of arbitrary sizes. The complex free algebra $\mathbb{C}\langle x\rangle$ over noncommuting variables $x=(x_1,\dots,x_d)$ comes with a natural involution $*$ that fixes x_j. This involution also naturally extends to matrices over $\mathbb{C}\langle x\rangle$. A noncommutative polynomial $f\in M_\delta(\mathbb{C}\langle x\rangle)$ is hermitian if $f^*=f$. The central geometric object of FRAG is the **positivity domain** of such an f,

$$\mathcal{D}_f=\bigcup_{n\in\mathbb{N}}\mathcal{D}_f(n),\qquad \mathcal{D}_f(n)=\{X\in H_n(\mathbb{C})^d\colon f(X)\succ 0\}$$

where $H_n(\mathbb{C})$ denotes the real space of $n\times n$ hermitian matrices. For the sake of normalization let $0\in\mathcal{D}_f$ be a quiet assumption from hereon; that is, $f(0)\succ 0$. Matricial sets of the form $\mathcal{D}_f$ are also called **(basic open) free semialgebraic sets**. They naturally appear in free analysis, operator systems and algebras, control and systems theory, relaxation schemes for polynomial optimization, and quantum information theory. Most of the arising questions concern convexity. Here $\mathcal{D}_f$ is **convex** if $\mathcal{D}_f(n)$ is a convex subset of $H_n(\mathbb{C})^d$ for every $n\in\mathbb{N}$.

Convex free semialgebraic sets have exceptional structural features in comparison with their classical cousins. An apparent example of a convex free semialgebraic set is a **free spectrahedron** $\mathcal{D}_L$, or a **linear matrix inequality (LMI) domain**, where $L=I+A_1x_1+\cdots+A_dx_d$ with $A_j\in H_\delta(\mathbb{C})$ is a monic hermitian pencil (an LMI representation of $\mathcal{D}_L$). It turns out [7] that every convex free semialgebraic set is a free spectrahedron. Furthermore, free spectrahedra admit a perfect Positivstellensatz [3]: a noncommutative polynomial f is positive semidefinite on $\mathcal{D}_L$ if and only if it belongs to the quadratic module generated by L, i.e.,

$$f=s_1^*s_1+\cdots+s_\ell^*s_\ell+s_{\ell+1}^*Ls_{\ell+1}+\cdots+s_m^*Ls_m$$

and $2\deg s_i\leq\deg f$. These two results have profound consequences for optimization over convex free semialgebraic sets. Namely, such optimization problems can be formulated as semidefinite programs, which can be efficiently solved using interior point methods with various implementations in computational software.

It is thus natural to ask how to check whether a free semialgebraic set is convex, and how to find its LMI representation if that is the case. It is worth pointing out that neither the original functional-analytic proof of the existence of an LMI representation nor the subsequent real-algebraic proofs are constructive. Fortunately, invariant and representation theory entered the picture in the last few years via the **free locus** of f,

$$\mathcal{Z}_f=\bigcup_{n\in\mathbb{N}}\mathcal{Z}_f(n),\qquad \mathcal{Z}_f(n)=\{X\in M_n(\mathbb{C})^d\colon \det f(X)=0\}.$$

One should think of $\mathcal{Z}_f$ as of the "Zariski closure" of the boundary of $\mathcal{D}_f$. In [8, 6] it was shown that persistent (as n grows) irreducible components of $\mathcal{Z}_f$ are in one-to-one correspondence with certain equivalence classes of factors of f over $\mathbb{C}\langle x\rangle$. In particular, f is irreducible if and only if $\mathcal{Z}_f(n)$ is a reduced irreducible hypersurface in $M_n(\mathbb{C})^d$ for large enough n. Together with the realization theory for noncommutative rational functions, these results were crucial for procedural study of semialgebraic convexity. Namely, in [5] an efficient algorithm (based on linear algebra, probabilistic methods and semidefinite programming) was designed for checking convexity of $\mathcal{D}_f$, and constructing its LMI representation. The derived machinery also has surprising theoretical consequences. Firstly, the intersection of a finite family of free semialgebraic sets with irreducible boundaries is convex if and only if each member of the family is convex. Secondly, if $f \in \mathbb{C}\langle x\rangle$ is hermitian and irreducible, and $\mathcal{D}_f$ is proper and convex, then f is a concave quadratic. Moreover, if f is hermitian and $\mathcal{D}_{f+\varepsilon}$ is proper and convex for all small enough $\varepsilon > 0$, then f is a composite of a univariate polynomial with a concave quadratic [10]. Similar methods were also used to analyze free stability on the matricial positive orthant and to prove the existence of determinantal representations for Hurwitz stable noncommutative polynomials and rational functions [9].

The success with convexity indicates a natural future quest: devise a (computationally efficient) procedure that determines which non-convex free semialgebraic sets can be analytically transformed into convex ones. Such a procedure would have important consequences for optimization in control theory, as it would accept a "hard" (non-convex) problem and return an equivalent "easy" (convex) problem. There are two kinds of partial results towards this goal. By [2], $\mathcal{D}_f$ admits a proper noncommutative rational map into a free spectrahedron if and only if there is a **plurisubharmonic** noncommutative rational function r such that $\mathcal{D}_f = \mathcal{D}_{-r}$. This gives a geometric condition for being transformable into a convex set. On the other hand, any deterministic non-convex–to–convex procedure would ostensibly rely on the output being rather unique: that is, one would hope that there are not many analytic maps between free spectrahedra. This is indeed true at least for spectrahedra with certain genericity assumptions [1] or symmetries [4]. Roughly speaking, if there is a bianalytic map $f : \mathcal{D}_L \to \mathcal{D}_M$ between two free spectrahedra (with the required features), then f is actually a **convexotonic map** (a birational map with lots of structure), and there is a pair of unitaries that intertwines the coefficients of the monic hermitian pencils L and M. The future research in FRAG will likely focus on extending these results to arbitrary free semialgebraic sets.

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