QUANTUM MAX d-CUT VIA QUDIT SWAP OPERATORS

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Abstract. Quantum Max Cut (QMC) problem for systems of qubits is an example of a 2-local Hamiltonian problem, and a prominent paradigm in computational complexity theory. This paper investigates the algebraic structure of a higher-dimensional analog of the QMC problem for systems of qudits. The Quantum Max d-Cut (d-QMC) problem asks for the largest eigenvalue of a Hamiltonian on a graph with n vertices whose edges correspond to swap operators acting on $(\mathbb{C}^d)^{\otimes n}$. The algebra generated by the swap operators is identified as a quotient of a free algebra modulo symmetric group relations and a single additional relation of degree d. This presentation leads to a tailored hierarchy of semidefinite programs, leveraging noncommutative polynomial optimization (NPO) methods, that converges to the solution of the d-QMC problem. For a large class of complete bipartite graphs, exact solutions for the d-QMC problem are derived using the representation theory of symmetric groups. This in particular includes the d-QMC problem for clique and star graphs on n vertices, for all d and n. Lastly, the paper addresses a refined d-QMC problem focused on finding the largest eigenvalue within each isotypic component (irreducible block) of the graph Hamiltonian. It is shown that the spectrum of the star graph Hamiltonian distinguishes between isotypic components of the 3-QMC problem. For general d, low-degree relations for separating isotypic components are presented, enabling adaptation of the global NPO hierarchy to efficiently compute the largest eigenvalue in each isotypic component.

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1. Introduction

The local Hamiltonian problem is a renowned problem in quantum computational complexity theory. It involves determining the largest (or smallest) eigenvalue of a given self-adjoint matrix H. The input matrix H acts on a space of n qubits and is hence of size $2^n \times 2^n$. It is expressed as a sum of *local* terms, i.e., for a chosen $k \leq n$,

$$H = \sum_{\substack{S \subseteq \{1,\dots,n\} \\ |S|=k}} H_S.$$

Here each H_S acts nontrivially only on a subset (of S) of at most k qubits. Such an H is called a k-local Hamiltonian.

The general k-local Hamiltonian problem is hard to solve; in fact, it belongs to the Quantum Merlin Arthur (QMA)-hard complexity class [KSV02, KKR06], which is a quantum analog of the NP-hard class. Hence, it is easier to approach by considering its specific instances, either by computing exact arithmetic solutions [LM62] or designing efficient (polynomial-time) high-precision algorithms to approximate the largest eigenvalue [LVV15]. Additional work was done on approximating the maximum eigenvalue up to a constant factor [GK12, BH13, BGKT19, HM17], and exploring hardness of computing ground space properties [GH24+].

We investigate generalizations of the Quantum Max Cut (QMC) problem, which is a special instance of the 2-local Hamiltonian problem, and was named by Gharibian and Parekh [GP19] as a quantum analog of the classical Max Cut problem for the Ising model (Section 1.1). The QMC problem naturally arises in physics as it seeks the ground state energy of the anti-ferromagnetic Heisenberg model for a system of interacting particles. The latter is used to describe magnetic properties of insular crystals, under the assumption that only the interactions of neighbor electrons in a lattice are significant (2-locality) [Aue94, BDZ08]. The QMC problem has recently become popular within the field of computational complexity theory. It is a simple prototype of a QMA-complete problem [PM17] and can hence be used for designing approximation algorithms to solve other QMA-hard problems [AMG20, PT21, PT22, Lee22, Kin23]. Arithmetic solutions to the QMC problem are known for certain families of graphs, such as complete bipartite graphs [LM62] and one-dimensional chains [LM16]. More recently, second order cone relaxations of the QMC capable of providing approximations for large graphs were introduced [HTPG24+].

The main objects used to define the QMC Hamiltonian are the **Pauli matrices**

$$\text{(Pauli)} \qquad \sigma_X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_Y = \begin{bmatrix} 0 & -\mathfrak{i} \\ \mathfrak{i} & 0 \end{bmatrix}, \quad \sigma_Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Together with the identity $\sigma_I := I$, they form a basis for $M_2(\mathbb{C})$. For fixed n let

$$\sigma_W^k = \underbrace{I_2 \otimes \cdots \otimes I_2}_{k-1} \otimes \sigma_W \otimes \underbrace{I_2 \otimes \cdots \otimes I_2}_{n-k} \in M_2(\mathbb{C})^{\otimes n} = M_{2^n}(\mathbb{C})$$

with $W \in \{X, Y, Z\}$ and $k \in \mathbb{N}$. Now

$$\{\sigma_{W_1}^1 \sigma_{W_2}^2 \cdots \sigma_{W_n}^n \mid W_j \in \{I, X, Y, Z\}\} \tag{1.1}$$

is a basis for $M_{2^n}(\mathbb{C})$.

A QMC Hamiltonian pertains to a given graph G on say n vertices. We denote by $\mathrm{V}(G)$ the vertex set of G and by $\mathrm{E}(G)$ the edge set of G.

Definition 1.1. Let G be a graph on n vertices and edge weights $\{w_{ij} \mid (i,j) \in \mathrm{E}(G)\}$. The Quantum Max Cut (QMC) Hamiltonian is defined as

$$H_G = \sum_{(i,j)\in \mathrm{E}(G)} w_{ij}\big(I - \sigma_X^i\sigma_X^j - \sigma_Y^i\sigma_Y^j - \sigma_Z^i\sigma_Z^j\big) \in M_{2^n}(\mathbb{C})_{\mathrm{sa}}. \tag{H_G}$$

The Quantum Max Cut (QMC) problem is about finding the largest eigenvalue of the QMC Hamiltonian H_G; that is, the ground state energy of $-H_G$.

1.1. **Connection to the classical Max Cut.** The QMC problem is named after the classical Max Cut (MC) problem [BPT13] of partitioning the vertices of a given graph into two sets such that the number or weight of the edges between the two sets is maximized. Equivalently, if the given graph G has edge set $\mathrm{E}(G)$ and edge weights $w_{ij} \geq 0$, maximize

$$\sum_{(i,j)\in \mathrm{E}(G)} w_{ij}\frac{1 - x_i x_j}{2}$$

over all possible evaluations at $x_i \in \{\pm 1\}$. Note that the MC problem is equivalent to the "diagonal" modification of the QMC problem, where the $\sigma_X^i\sigma_X^j$ and $\sigma_Y^i\sigma_Y^j$ terms in (H_G) are dropped. Alternatively, while the QMC problem seeks the ground state energy of the Heisenberg XXX model, classical MC problem seeks the ground state energy of the Ising model (without an external field).

Solving the MC problem in general is NP-hard, thus several approximation algorithms were developed. The most famous approximation algorithm is by Goemans and Williamson [GW95], and is based on semidefinite programming (SDP) [BPT13]. It can be understood as the first level of Lasserre's Moment-SOS (Sum-of-Squares) hierarchy of SDP relaxations [Lse01] (see also [Lau09, HKL20, Nie23]) that give a converging sequence of upper bounds to the exact solution of the MC problem. Raghavendra [Rag08, Rag09] showed, assuming the Unique Games Conjecture of Khot [Kho02], that no polynomial-time algorithm for the MC problem is better than the Goemans-Williamson algorithm (unless P=NP).

1.2. **Quantum Max Cut.** To tackle the QMC problem, the algebraic structure of the QMC Hamiltonian is investigated in [BCEHK24, TRZ+]. This approach starts by rephrasing H_G in terms of the swap matrices Swap_{ij}.[1]

Definition 1.2. For fixed n and $1 \leq i < j \leq n$, the swap matrix $\mathrm{Swap}_{ij} \in M_{2^n}(\mathbb{C})$ is defined by sending any rank one tensor

$$v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v_n \in (\mathbb{C}^2)^{\otimes n}$$

to the rank one tensor

$$v_1 \otimes \cdots \otimes v_j \otimes \cdots \otimes v_i \otimes \cdots \otimes v_n \in (\mathbb{C}^2)^{\otimes n},$$

where $v_k \in \mathbb{C}^2$.

One can directly compute that any swap matrix Swap_{ij} is expressed in terms of the Pauli matrices as

$$\mathrm{Swap}_{ij} = \frac{1}{2}(I + \sigma_X^i\sigma_X^j + \sigma_Y^i\sigma_Y^j + \sigma_Z^i\sigma_Z^j). \tag{1.2}$$

[1] Physics literature often calls these SWAP or exchange operators [NC10].

Using (1.2), the QMC Hamiltonian (H_G) can be expressed in terms of the swap matrices rather than the Pauli matrices (1.1).

Proposition 1.3. *The QMC Hamiltonian from* (H_G) *is given in terms of the swap matrices* Swap_{ij} *as*

$$H_G = \sum_{(i,j)\in \mathrm{E}(G)} 2w_{ij}(I - \mathrm{Swap}_{ij}). \tag{1.3}$$

1.3. **Swap matrices on general qudit spaces.** In this article we consider the QMC problem on qudits instead of qubits. As qudits store more information than qubits, systems of interacting qudits are a natural framework for quantum computing with less resources [WHSK20]. Here, the swap matrices $\mathrm{Swap}_{ij}^{(d)}$ act on $(\mathbb{C}^d)^{\otimes n}$ for some $d \geq 2$. In analogy with the $d = 2$ case, they act as transpositions on n-qudit states.

Definition 1.4. For fixed n and $1 \leq i < j \leq n$, the (**qudit**) **swap matrix** $\mathrm{Swap}_{ij}^{(d)}$ is defined by its action on rank one tensors as

$$\mathrm{Swap}_{ij}^{(d)}(v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v_n) = v_1 \otimes \cdots \otimes v_j \otimes \cdots \otimes v_i \otimes \cdots \otimes v_n$$

for any $v_1, \dots, v_n \in \mathbb{C}^d$.

The action of swap matrices on qudits yields a representation $\rho_n^{(d)}$ of S_n on $(\mathbb{C}^d)^{\otimes n}$ defined by

$$\rho_n^{(d)}(\pi)(v_1 \otimes \cdots \otimes v_n) = v_{\pi^{-1}(1)} \otimes \cdots \otimes v_{\pi^{-1}(n)}.$$

We denote the image $\rho_n^{(d)}(\mathbb{C}[S_n])$, which is a subalgebra of $M_{d^n}(\mathbb{C})$, by $M_n^{\mathrm{Sw}_d}(\mathbb{C})$. It is called the d-**swap algebra**. Guided by the expression (1.3) of the QMC Hamiltonian in terms of the swap matrices, one can define the Quantum Max d-Cut Hamiltonian via the qudit swap matrices $\mathrm{Swap}_{ij}^{(d)}$.

Definition 1.5. Let G be a graph on n vertices and edge weights $\{w_{ij} \mid (i,j) \in \mathrm{E}(G)\}$. The **Quantum Max** d**-Cut** (d**-QMC**) **Hamiltonian** is defined as

$$H_G^d = \sum_{(i,j)\in \mathrm{E}(G)} 2w_{ij}\left(I - \mathrm{Swap}_{ij}^{(d)}\right). \tag{H_G^d}$$

The d**-QMC problem** again asks for the largest eigenvalue of the d-QMC Hamiltonian H_G^d in (H_G^d). The problem is motivated by determining ground state energies of $\mathrm{SU}(d)$-Heisenberg models on lattices [KT07, BAMC09, PM21]. While the QMC problem is the quantum analog of the classical MC problem, the d-QMC problem is the quantum analog of the d-MC problem pertaining to maximal d-colorable subgraphs, and the anti-ferromagnetic d-state Potts model [FJ97]. The d-QMC problem was first considered in this context by [CJKKW+] in 2023. There the authors define the d-QMC Hamiltonian with the use of the *Gell-Mann matrices*, which are a generalization of the Pauli matrices to any size $d \times d$. We give more insight into this approach and show that is equivalent to ours in Section 1.5 below.

In order to develop an algebraic toolbox for solving the d-QMC problem, it is essential to determine the precise relations that define the d-swap algebra. Since the transpositions

(i,j) generate S_n, the swap matrices generate $M_n^{\mathrm{Sw}_d}(\mathbb{C})$. Hence, similar to the transpositions, for distinct indices i,j,k,l, the swap matrices satisfy the relations

$$\begin{aligned} (\mathrm{Swap}_{ij}^{(d)})^2 &= 1, \\ \mathrm{Swap}_{ij}^{(d)}\,\mathrm{Swap}_{kl}^{(d)} &= \mathrm{Swap}_{kl}^{(d)}\,\mathrm{Swap}_{ij}^{(d)}, \\ \mathrm{Swap}_{ij}^{(d)}\,\mathrm{Swap}_{jk}^{(d)} &= \mathrm{Swap}_{ik}^{(d)}\,\mathrm{Swap}_{ij}^{(d)} = \mathrm{Swap}_{jk}^{(d)}\,\mathrm{Swap}_{ik}^{(d)}. \end{aligned} \tag{1.4}$$

For $d=2$, it is known (see [BCEHK24, Theorem 3.6] and [TRZ+, Theorem 3.8], and [Pro07, Theorem 11.6.1] for general d) that the swap matrices additionally satisfy the **degree-reducing relation**

$$\mathrm{Swap}_{ij}^{(2)}\,\mathrm{Swap}_{jk}^{(2)} + \mathrm{Swap}_{jk}^{(2)}\,\mathrm{Swap}_{ij}^{(2)} = \mathrm{Swap}_{ij}^{(2)} + \mathrm{Swap}_{jk}^{(2)} + \mathrm{Swap}_{ik}^{(2)} - 1, \tag{1.5}$$

and that the symmetric group relations (1.4) together with the degree-reducing relation (1.5) precisely define $M_n^{\mathrm{Sw}_2}(\mathbb{C})$. In Section 3 we show that the general swap matrices $\mathrm{Swap}_{ij}^{(d)}$ are also characterized by a (slightly more complicated) degree-reducing relation; see Proposition 3.1.

1.4. **Main results.**

This paper applies the representation theory of the symmetric group S_n to explore and take advantage of the algebraic structure and symmetries inherent to the d-QMC problem. Throughout the text, we refer to irreducible representations as **irreps**. Our contributions are as follows.

1.4.1. *Defining relations of the d-swap algebra.*

In Section 4, we identify the d-swap algebra $M_n^{\mathrm{Sw}_d}(\mathbb{C})$ as a quotient of the free algebra generated by the $\binom{n}{2}$ freely noncommuting variables swap_{ij} for $1 \le i < j \le n$. For $k \in \mathbb{N}$ denote

$$c_k = \sum_{\substack{1\le i_0,\dots,i_k\le d \\ \text{pairwise distinct,} \\ i_0<i_j \text{ for } j\ge 1}} \mathrm{swap}_{i_0 i_1}\mathrm{swap}_{i_0 i_2}\cdots\mathrm{swap}_{i_0 i_k}. \tag{1.6}$$

Theorem 4.3 below states that $M_n^{\mathrm{Sw}_d}(\mathbb{C})$ is isomorphic to the quotient of the free algebra $\mathbb{C}\langle \mathrm{swap}_{ij} \colon 1 \le i < j \le n\rangle$ modulo the relations

$$\begin{aligned} \mathrm{swap}_{ij}^2 &= 1, \\ \mathrm{swap}_{ij}\mathrm{swap}_{jk} &= \mathrm{swap}_{ik}\mathrm{swap}_{ij} = \mathrm{swap}_{jk}\mathrm{swap}_{ik}, \\ \mathrm{swap}_{ij}\mathrm{swap}_{kl} &= \mathrm{swap}_{kl}\mathrm{swap}_{ij}, \\ c_d &= c_{d-1} - c_{d-2} + \cdots + (-1)^{d-1}c_1 + (-1)^d. \end{aligned} \tag{1.7}$$

We acknowledge that this isomorphism may not be new to experts in representation theory, who will recognize the last equation in (1.7) as the vanishing of an antisymmetrizer of $d+1$ vectors on $(\mathbb{C}^d)^{\otimes n}$. Nevertheless, in Section 3 we provide an elementary and self-contained proof that the last relation in (1.7) completely determines $M_n^{\mathrm{Sw}_d}(\mathbb{C})$ as the quotient of the group algebra of S_n. To achieve this, the Schur-Weyl duality is invoked to assess the precise decomposition of $M_n^{\mathrm{Sw}_d}(\mathbb{C})$ into irreps, as follows.

Theorem 2.2. *The d-swap algebra $M_n^{\mathrm{Sw}_d}(\mathbb{C})$ decomposes into a direct sum of simple algebras generated by the irreps ρ_λ of S_n corresponding to partitions of n with at most d*

rows,

$$M_n^{\mathrm{Sw}_d}(\mathbb{C}) \cong \bigoplus_{\substack{\lambda\vdash n\\ \mathrm{ht}(\lambda)\le d}} \rho_\lambda(\mathbb{C}S_n).$$

1.4.2. *NPO hierarchy for the d-QMC problem.* A widely used approach for solving local Hamiltonian problems is through semidefinite programming (SDP) relaxations and noncommutative polynomial optimization (NPO) [NPA08, PNA10, DLTW08, BKP16]. While the d-QMC problem is already an SDP of the form

$$\max_\rho \mathrm{tr}(\rho H_G^d) \quad \text{subject to } \rho\succeq 0 \text{ and } \mathrm{tr}(\rho)=1,$$

this formulation is hopeless for large n because the semidefinite constraint is a $d^n\times d^n$ matrix. Instead, one needs to explore the 2-locality of the d-QMC problem. As in the case of the classical MC, one can define a hierarchy of SDP relaxations which can be computed efficiently, i.e., in polynomial time, and give upper bounds to the true maximum eigenvalue of H [BH13, BGKT19, GP19, PT21, HO22]. However, due to the exponential growth of the size of the matrices, only the first few levels are tractable.

Having identified the d-swap algebra as a quotient of the free algebra $\mathbb{C}\langle \mathrm{swap}_{ij}\rangle$ in Section 4, the d-QMC problem is written as a more efficient instance of a NPO problem in Section 5. The d-QMC Hamiltonian H_G^d is represented by an element $h_G\in\mathbb{C}\langle \mathrm{swap}_{ij}\rangle$, and its largest eigenvalue is

$$\alpha_* = \min\left\{\alpha : \alpha - h_G \text{ is a sum of hermitian squares in } \mathbb{C}\langle \mathrm{swap}_{ij}\rangle \text{ modulo (1.7)}\right\}.$$

By adapting the non-commutative Sum-of-Squares hierarchy (ncSoS) from [BCEHK24], we give a sequence of semidefinite programs (SDPs) whose solutions approximate α_* from above. This scheme is specifically tailored to the algebraic structure of the swap matrices defining the d-QMC Hamiltonian. Since the d-swap algebra satisfies the symmetric group relations, this hierarchy is exact at level $n-1$. For large graphs G, only the first few levels of the hierarchy are practical for computations. For this reason we also focus on low-degree relations of swap matrices, which play a role in the construction of the SDPs for the first two levels of the hierarchy. Appendices A and B provide explicit bases for products of swap operators of low degree.

1.4.3. *Exact solutions for cliques and star graphs.* In Section 6 we turn our attention to computing the exact solutions to the d-QMC problem for certain families of graphs. To achieve this, we explore the isotypic structure of d-QMC Hamiltonians. Given a partition $\lambda\vdash n$, the λ-block of a d-QMC Hamiltonian is its isotypic component corresponding to ρ_λ under the isomorphism of Theorem 2.2 above. The d-QMC problem for cliques K_n on n vertices is easiest to address as the isotypic blocks of the corresponding d-QMC Hamiltonian are scalar matrices (see Lemma 6.4). Let η_λ denote the eigenvalue of the block corresponding to the partition λ. The following theorem gives an explicit expression of λ in terms of its rows $\lambda_1,\dots,\lambda_d$, and identifies the partition λ that maximizes η_λ; i.e., the solution to the d-QMC problem for an n-clique K_n is computed.

Theorem 1.6. *For any $\lambda\vdash n$ with rows $\lambda_1\ge\cdots\ge\lambda_d\ge 1$,*

$$\eta_\lambda = n^2 + \frac{d(d-1)(2d-1)}{6} - \sum_{k=1}^{d}\big(\lambda_k-(k-1)\big)^2.$$

The maximum value of η_λ among all partitions $\lambda \vdash n$ with $\mathrm{ht}(\lambda) \le d$ is obtained at

$$\lambda = \Big(\underbrace{1+\frac{n-r}{d},\dots,1+\frac{n-r}{d}}_{r},\ \underbrace{\frac{n-r}{d},\dots,\frac{n-r}{d}}_{d-r}\Big)$$

for $n \equiv r \pmod d$. The solution to the d-QMC problem for an n-clique hence equals

$$n^2 + (d-1)n + r^2 - r(d+1) - \frac{n^2-r^2}{d}.$$

For the proof of Theorem 1.6 see Proposition 6.7 and Corollary 6.8.

To tackle the d-QMC problem on a more general class of graphs, we use a principle from [BCEHK24] called *clique decomposition*. It is a way of writing the d-QMC Hamiltonian of a given graph as an alternating sum of Hamiltonians of cliques and simpler graphs. A graph with a simple clique decomposition is the star graph $\bigstar_n$ on n vertices, on which we focus in Section 6.3.1. The relation

$$\bigstar_n = K_n - K_{n-1}$$

holds, where the minus sign means deleting from K_n the edges that appear in K_{n-1}. This decomposition was used in [BCEHK24] to show that the solution to the 2-QMC problem for the star graph $\bigstar_n$ is $2n$, attained at the partition $\lambda = (n-1, 1)$. Extending this result, we solve the d-QMC problem for $\bigstar_n$.

Theorem 6.11. *If $\lambda = (\lambda_1, \dots, \lambda_d) \vdash n$ has d rows $\lambda_1 \ge \cdots \ge \lambda_d$, then the eigenvalues of the λ-block of the d-QMC Hamiltonian $H^d_{\bigstar_n}$ form a subset of*

$$\{2(n-\lambda_1),\ 2(n-\lambda_2+1),\ \dots,\ 2(n-\lambda_d+d-1)\}$$

containing the value $\eta_\star = 2(n-\lambda_d+d-1)$. Hence, the solution to the d-QMC problem for $\bigstar_n$ is $2(n+d-2)$, obtained by plugging $\lambda_d = 1$ into $2(n-\lambda_d+d-1)$.

1.4.4. *Exact solutions for complete bipartite graphs.* Star graphs are special cases of complete bipartite graphs. In Section 6.3.2, we use the clique decomposition to exactly solve the d-QMC problem for general complete bipartite graphs $K_{n-k,k}$ (where without loss of generality $n-k \ge k$) as long as d is sufficiently small.

Theorem 6.16. *Let $k \le \frac{n}{2}$. If $d < \frac{n}{k-1}$, then the solution to the d-QMC problem for $K_{n-k,k}$ is*

$$\begin{cases} 2(n+d)k - \left(\left\lfloor\sqrt{k}\right\rfloor + 1\right)\left(2k - \left\lfloor\frac{k}{\lfloor\sqrt{k}\rfloor}\right\rfloor \lfloor\sqrt{k}\rfloor\right)\left(1 + \left\lfloor\frac{k}{\lfloor\sqrt{k}\rfloor}\right\rfloor\right) & \text{if } d \ge 1 + \lfloor\sqrt{k}\rfloor, \\ 2(n+d)k - d\left(2k - \lfloor\frac{k}{d-1}\rfloor(d-1)\right)\left(1 + \lfloor\frac{k}{d-1}\rfloor\right) & \text{if } d < 1 + \lfloor\sqrt{k}\rfloor. \end{cases}$$

Moreover, we identify partitions λ at which the solution values in Theorem 6.16 are attained. Note that the d-QMC problem for $K_{n-1,1}$ (the star graphs) and $K_{n-2,2}$ (Example 6.26) is resolved for every $d < n$ by Theorem 6.16 (the case $d \ge n$ is simple, see Proposition 6.3). Furthermore, Example 6.27 solves the d-QMC for $K_{n-3,3}$ for every $d < n$, while Example 6.28 demonstrates the limitation of Theorem 6.16 for $K_{n-4,4}$.

1.4.5. *Separation of irreps.* In Section 7, we refine the NPO hierarchy for the d-QMC problem, with the intention of calculating the maximum eigenvalue of the λ-block in H^d_G corresponding to a given irrep of S_n given by the partition $\lambda \vdash n$. The idea is to find low-degree polynomials that distinguish distinct irreps indexed by partitions with at most

d rows. In Theorem 7.4 we show that irreps of S_n are separated by the polynomials c_k of degree k from (1.6). This leads to an NPO formulation of the localized d-QMC problem.

Theorem 1.7. *Let $\lambda \vdash n$ have at most d rows. There are constants $\gamma_1, \dots, \gamma_{d-1} \in \mathbb{Z}$ such that the largest eigenvalue of the λ-block in H_G^d equals*

$$\min \big\{ \alpha : \alpha - h_G \text{ is a sum of hermitian squares in } \mathbb{C}\langle \mathrm{swap}_{ij} \rangle \\ \text{modulo (1.7) and } c_1 = \gamma_1, \dots, c_{d-1} = \gamma_{d-1} \big\}.$$

The NPO problem in Theorem 1.7 can be tackled with an NPA-like hierarchy of SDPs, and the values γ_k can be evaluated using explicit Lassalle's character formulas for cycles in S_n [Lsa08].

We also consider distinguishability of irreps of S_n from the perspective of the d-QMC problem. As d-QMC Hamiltonians can only admit λ-blocks for $\lambda \vdash n$ with at most d rows, one can only hope to distinguish such irreps through the d-QMC problem. For $d = 2$, it is known that the value η_λ itself separates irreps with at most two rows [BCEHK24]; in other words, the spectrum of the Hamiltonian for K_n separates irreps with at most two rows. This is not the case anymore for $d > 2$. However, we show that for $d = 3$, the spectrum of the 3-QMC Hamiltonian for $\bigstar_n$ separates irreps with at most three rows.

Theorem 1.8. *Let $\lambda, \mu \vdash n$ be partitions with at most three rows. Then $\lambda = \mu$ if and only if the spectra of the λ-block and the μ-block of $H^3_{\bigstar_n}$ coincide.*

See Proposition 7.2 for the proof. For general $d, n \in \mathbb{N}$, we leave it as an open problem whether there exist graphs $G_1, \dots, G_\ell$ on n vertices such that for all $\lambda, \mu \vdash n$ with at most d rows, $\lambda = \mu$ if and only if the spectra of the λ-block and the μ-block of $H^d_{G_i}$ coincide for all $i = 1, \dots, \ell$.

1.5. **Comparison with the work of Carlson, Jorquera, Kolla, Kordonowy, Wayland.** The d-QMC problem was introduced in [CJKKW+], where it was defined via a generalization of the Pauli matrices to arbitrary size $d \times d$, called Gell-Mann matrices.

1.5.1. *The Gell-Mann matrices.* For each $d \geq 2$, there is a family of $d^2 - 1$ trace zero self-adjoint matrices, which, together with the identity I_d, form a basis for $M_d(\mathbb{C})$. More concretely, for $d = 2$ these are the Pauli matrices, and for $d \geq 3$ there are three kinds of $d \times d$ Gell-Mann matrices (see Appendices B.2 and B.5, where these matrices are given explicitly for $d = 3$ and $d = 4$, respectively):

$$\begin{aligned}
&\text{symmetric:} && \lambda_{a,b}^{\mathrm{sym}_d} = E_{a,b} + E_{b,a}, && 1 \leq a < b \leq d, \\
&\text{antisymmetric:} && \lambda_{a,b}^{\mathrm{asym}_d} = \mathfrak{i}(E_{b,a} - E_{a,b}), && 1 \leq a < b \leq d, \\
&\text{diagonal:} && \lambda_k^d = \lambda_k^{d-1} \oplus 0, && 2 \leq k < d, \\
& && \lambda_d^d = \sqrt{\frac{2}{d(d-1)}} \big(I_{d-1} \oplus (1-d) \big). &&
\end{aligned} \tag{1.8}$$

Here, $E_{a,b}$ are standard matrix units, and I_{d-1} is the identity of size $d-1$. Note that there are $\binom{d}{2}$ (non-diagonal) symmetric, $\binom{d}{2}$ antisymmetric and $d-1$ diagonal matrices. Summing up we get $2\binom{d}{2} + d - 1 = d(d-1) + d - 1 = d^2 - 1$ as expected. For fixed d denote by $GM(d)$ the set consisting of the $d^2 - 1$ Gell-Mann $d \times d$ matrices of size $d \times d$, together with the identity I_d.

As for the Pauli matrices, define for a fixed $n \in \mathbb{N}$,

$$\lambda^j := \underbrace{I \otimes \cdots \otimes I}_{j-1} \otimes \lambda \otimes I \otimes \cdots \otimes I \in M_{d^n}(\mathbb{C})$$

for any $\lambda \in GM(d)$. By definition, λ_1^i and λ_2^j commute for any $i \neq j$ and $\lambda_1, \lambda_2 \in GM(d)$, and

$$\{\lambda_1^1 \lambda_2^2 \cdots \lambda_n^n \mid \lambda_j \in GM(d),\ j = 1, \dots, n\} \tag{1.9}$$

is a basis of $M_{d^n}(\mathbb{C})$.

1.5.2. *The d-QMC Hamiltonian via the Gell-Mann matrices.* The formula (1.2) expressing the swap matrices $\mathrm{Swap}_{ij}^{(2)}$ in terms of products of Pauli matrices, i.e., with respect to the basis (1.9) for $d = 2$, can be generalized to an arbitrary d as shown below.

Proposition 1.9. *For any $i < j$ we have*

$$\mathrm{Swap}_{ij}^{(d)} = \frac{1}{d} I + \frac{1}{2} \sum_{\lambda \in GM(d)} \lambda^i \lambda^j. \tag{1.10}$$

Proof. Denote the right-hand side of (1.10) by R. Since (1.10) only involves two indices i, j we may assume $n = 2$ and prove that R acts the same as $\mathrm{Swap}_{ij}^{(d)}$ on basis vectors of the form $v_{p,q} = e_p \otimes e_q \in \mathbb{C}^d \otimes \mathbb{C}^d$ for $1 \leq p, q \leq d$.

If $p = q$, then only the diagonal Gell-Mann matrices λ_k^d for $p \leq k \leq d$ act nontrivially on $v_{p,q}$,

$$\begin{aligned} Rv_{p,p} &= \frac{1}{d} e_p \otimes e_p + \frac{1}{2} \left(\sqrt{\frac{2}{p(p-1)}} (1-p) e_p \right) \otimes \left(\sqrt{\frac{2}{p(p-1)}} (1-p) e_p \right) \\ &\quad + \frac{1}{2} \sum_{j=p+1}^{d} \left(\sqrt{\frac{2}{j(j-1)}} e_p \right) \otimes \left(\sqrt{\frac{2}{j(j-1)}} e_p \right) \\ &= \left(\frac{1}{d} + \frac{p-1}{p} + \sum_{j=p+1}^{d} \frac{1}{j(j-1)} \right) v_{p,p} = v_{p,p}. \end{aligned}$$

Finally, if $p < q$, then in addition to λ_k^d for $q \leq k \leq d$, also $\lambda_{p,q}^{\mathrm{sym}_d}$ and $\lambda_{p,q}^{\mathrm{asym}_d}$ act nontrivially on $v_{p,q}$ and they both map it to $v_{q,p}$,

$$\begin{aligned} Rv_{p,q} &= \frac{1}{d} e_p \otimes e_q + \frac{1}{2} \left(\sqrt{\frac{2}{q(q-1)}} e_p \right) \otimes \left(\sqrt{\frac{2}{q(q-1)}} (1-q) e_q \right) \\ &\quad + \frac{1}{2} \sum_{j=q+1}^{d} \left(\sqrt{\frac{2}{j(j-1)}} e_p \right) \otimes \left(\sqrt{\frac{2}{j(j-1)}} e_q \right) \\ &\quad + \frac{1}{2} e_q \otimes e_p + \frac{1}{2} e_q \otimes e_p \\ &= \left(\frac{1}{d} - \frac{1}{q} + \sum_{j=q+1}^{d} \frac{1}{j(j-1)} \right) v_{p,q} + v_{q,p} = v_{q,p}. \end{aligned}$$

■

Note that by (1.10), the d-QMC Hamiltonian can be expressed in terms of the $d \times d$ Gell-Mann matrices as

$$H_G^d = \sum_{(i,j)\in \mathrm{E}(G)} 2w_{ij} \left(\frac{d-1}{d} I - \frac{1}{2} \sum_{\lambda \in GM(d)} \lambda^i \lambda^j \right), \tag{1.11}$$

where $GM(d)$ denotes the set of all $d \times d$ Gell-Mann matrices. This is in fact the form of the d-QMC Hamiltonian used in [CJKKW+].

An advantage of our approach is that it incorporates algebraically *all* symmetries inherent to the d-QMC problem. Computations (such as the NPO relaxations in Section 5, the clique decomposition in Section 6 or the decompositions along irreps) with H_G^d as in (H_G^d) scale better with both n and d or are only made possible once one passes to qudit swap matrices championed in this paper.

2. Preliminaries on the representations of the symmetric group

In this section we review some standard elements of the representation theory of symmetric groups that are used throughout this paper; for a comprehensive source, see, e.g., [FH91, Pro07]. For $n \in \mathbb{N}$ we denote by S_n the symmetric group on n elements, i.e., the group of permutations of n elements. A **representation** of S_n is a group homomorphism $\rho : S_n \to \mathrm{GL}(V)$ for a vector space V, also called S_n-module. Any representation ρ of S_n naturally defines a representation $\tilde{\rho}$ of the **group algebra** $\mathbb{C}[S_n]$ of S_n. The resulting representation $\tilde{\rho} : \mathbb{C}[S_n] \to \mathrm{End}(V)$ is defined by

$$\tilde{\rho}\left(\sum_{\pi \in S_n} \alpha_\pi \pi \right) = \sum_{\pi \in S_n} \alpha_\pi \rho(\pi).$$

For simplicity, the letter ρ often refers to both the representation of S_n and the representation of $\mathbb{C}[S_n]$.

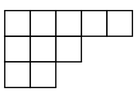
2.1. Irreducible representations of the symmetric group.

An S_n-module V is **irreducible** if its only nontrivial submodule is V. Throughout we abbreviate irreducible representation by *irrep* and use the terms irrep and irreducible module interchangeably. Note that by Maschke's theorem [Pro07, Section 6.1.5], any S_n-module V decomposes as a direct sum of irreducible S_n-modules.

It is well-known that the irreducible S_n-modules are indexed by partitions λ of n (often denoted by $\lambda \vdash n$), where

$$\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{N}^k, \quad \lambda_1 \geq \dots \geq \lambda_k > 0, \quad \sum_{i=1}^k \lambda_i = n.$$

The number of summands k is called the **height** of λ and denoted $k = \mathrm{ht}(\lambda)$. A partition $\lambda \vdash n$ is usually depicted by its **Young diagram**. A Young diagram of shape λ has k rows and the i-th row consists of λ_i boxes. For example, if $\lambda = (5,3,2) \vdash 10$, then $\mathrm{ht}(\lambda) = 3$ and its Young diagram is



A **Young tableau** of shape λ is a Young diagram whose boxes are filled with numbers $1, \dots, n$ such that each box gets a different integer. The symmetric group S_n acts on a Young tableau t of shape $\lambda \vdash n$ by permuting the entries of t. This action defines an equivalence relation, where two tableaux are equivalent if one can be obtained from the other by permuting the entries within each of the rows. Equivalence classes with respect to this relation are called **tabloids**.

The irreducible S_n-module V_λ corresponding to the partition $\lambda \vdash n$ is called a **Specht module** and it has a well-known basis consisting of **polytabloids**

$$e_T = \sum_{\pi \in C_T} \operatorname{sgn}(\pi)\pi\{T\}.$$

Here T ranges over all tabloids of shape λ, C_T is the set of all permutations that permute the elements only within the columns of T and for each $\pi \in C_T$, $\pi\{T\}$ is the tabloid obtained from T by permuting the entries according to π.

2.2. **Schur-Weyl duality.** As a complex representation of S_n, the d-swap algebra $M_n^{\mathrm{Sw}_d}(\mathbb{C})$ is semisimple. Key to solving the d-QMC problem for certain graphs is the precise knowledge of the block decomposition of $M_n^{\mathrm{Sw}_d}(\mathbb{C})$ into simple matrix algebras. We now explain how this block decomposition can be deduced using the Schur-Weyl duality of the actions of S_n and $\mathrm{GL}_d(\mathbb{C})$ on $(\mathbb{C}^d)^{\otimes n}$ (see e.g. [FH91, Section 6.1] or [Pro07, Section 7.1.2]). The natural representation of $\mathrm{GL}_d(\mathbb{C})$ on $(\mathbb{C}^d)^{\otimes n}$ is defined by the diagonal action; $g \in \mathrm{GL}_d(\mathbb{C})$ acts on the tensor product of $v_1, \dots, v_n \in \mathbb{C}^d$ by

$$\zeta_n(g)(v_1 \otimes \cdots \otimes v_n) = g(v_1) \otimes \cdots \otimes g(v_n).$$

The actions of S_n and $\mathrm{GL}_d(\mathbb{C})$ on $(\mathbb{C}^d)^{\otimes n}$ commute and there is a bijection between the irreducible modules of S_n and $\mathrm{GL}_d(\mathbb{C})$. This interplay between permutations of particles and change of coordinates is indispensable in investigating qudit systems, see e.g. [GNW21]. Furthermore, if λ is a partition of n, then to the irreducible module V_λ of S_n corresponds (up to isomorphism) exactly one irreducible module L_λ of $\mathrm{GL}_d(\mathbb{C})$ and L_λ are precisely the maps from V_λ to $(\mathbb{C}^d)^{\otimes n}$ that commute with the action of S_n,

$$L_\lambda = \mathrm{Hom}_{S_n}(V_\lambda, (\mathbb{C}^d)^{\otimes n}).$$

It is well-known that L_λ is nonzero precisely when λ is a partition with at most d rows [Pro07, Proposition 9.3.1]. In fact, the dimensions of the modules L_λ can be explicitly computed by the Weyl character formula [Pro07, Section 9.6.2].

The next proposition is a restatement of the Schur-Weyl duality [Pro07, Theorem 9.3.1] for S_n and $\mathrm{GL}_d(\mathbb{C})$, taking into account [Pro07, Proposition 9.3.1].

Proposition 2.1. *The algebras $M_n^{\mathrm{Sw}_d}(\mathbb{C})$ and $\zeta_n(\mathrm{GL}_d(\mathbb{C}))$ are centralizers of one another inside $\mathrm{End}((\mathbb{C}^d)^{\otimes n}) = M_{d^n}(\mathbb{C})$, and with respect to the action of the direct product $GL_d(\mathbb{C}) \times S_n$, the space $(\mathbb{C}^d)^{\otimes n}$ decomposes as*

$$(\mathbb{C}^d)^{\otimes n} \cong \bigoplus_{\substack{\lambda \vdash n \\ \mathrm{ht}(\lambda) \le d}} L_\lambda \otimes V_\lambda. \tag{2.1}$$

Since S_n acts trivially on each L_λ, the space $(\mathbb{C}^d)^{\otimes n}$ decomposes as an S_n-module into irreducible modules V_λ (with multiplicities) as follows

$$(\mathbb{C}^d)^{\otimes n} \cong \bigoplus_{\substack{\lambda \vdash n \\ \mathrm{ht}(\lambda) \le d}} V_\lambda^{\dim(L_\lambda)}.$$

As a corollary we get the desired decomposition of the d-swap algebra $M_n^{\mathrm{Sw}_d}(\mathbb{C})$.

Theorem 2.2. *The d-swap algebra decomposes into a direct sum of simple algebras generated by the irreps ρ_λ of S_n corresponding to partitions of n with at most d rows,*

$$M_n^{\mathrm{Sw}_d}(\mathbb{C}) \cong \bigoplus_{\substack{\lambda\vdash n \\ \mathrm{ht}(\lambda)\le d}} \rho_\lambda(\mathbb{C}S_n) \cong \bigoplus_{\substack{\lambda\vdash n \\ \mathrm{ht}(\lambda)\le d}} M_{\dim(V_\lambda)}(\mathbb{C}).$$

Proof. Using (2.1) we deduce that as a $\mathrm{GL}_d(\mathbb{C})$-module, the space $(\mathbb{C}^d)^{\otimes n}$ decomposes as

$$\left(\mathbb{C}^d\right)^{\otimes n} \cong \bigoplus_{\substack{\lambda\vdash n \\ \mathrm{ht}(\lambda)\le d}} L_\lambda^{\dim(V_\lambda)}.$$

Now considering the $\mathrm{GL}_d(\mathbb{C})$-endomorphisms on both sides gives the desired result. Indeed, the $\mathrm{GL}_d(\mathbb{C})$-endomorphisms of $(\mathbb{C}^d)^{\otimes n}$ are by definition the endomorphisms of $(\mathbb{C}^d)^{\otimes n}$ that commute with the action of $\mathrm{GL}_d(\mathbb{C})$. By the Schur-Weyl duality these are precisely the elements from the d-swap algebra $M_n^{\mathrm{Sw}_d}(\mathbb{C})$. On the other hand,

$$\begin{aligned}
\mathrm{End}_{\mathrm{GL}_d(\mathbb{C})}\left(\bigoplus_{\substack{\lambda\vdash n \\ \mathrm{ht}(\lambda)\le d}} L_\lambda^{\dim(V_\lambda)} \right) &= \bigoplus_{\substack{\lambda\vdash n \\ \mathrm{ht}(\lambda)\le d}} \mathrm{End}_{\mathrm{GL}_d(\mathbb{C})}\left(L_\lambda^{\dim(V_\lambda)}\right) \\
&= \bigoplus_{\substack{\lambda\vdash n \\ \mathrm{ht}(\lambda)\le d}} M_{\dim(V_\lambda)}\left(\mathrm{End}(L_\lambda)^{\mathrm{op}}\right) \\
&\cong \bigoplus_{\substack{\lambda\vdash n \\ \mathrm{ht}(\lambda)\le d}} M_{\dim(V_\lambda)}(\mathbb{C})
\end{aligned}$$

■

Remark 2.3. The dimension of any irreducible S_n-module V_λ can be computed via the well-known hook length formula (see Section 6 for some explicit calculations).

3. Degree-reducing relation for qudit swap matrices

It is known (see, e.g. [Pro07, Section 9.3]) that, in addition to the symmetric group axioms, the swap matrices $\mathrm{Swap}_{ij}^{(d)}$ also satisfy a degree-reducing relation of degree d. For instance, (1.5) above is such an equation for $d = 2$. We now present the general form of the degree-reducing relation for general d, and, for the reader's convenience, give an elementary and self-contained proof.

Since the symmetric group is generated by transpositions, each permutation can be written as a product of transpositions (non-uniquely). For fixed d and $k = 1, \dots, d$ let C_k be the set of all products of k swap matrices that arise from permutations on a subset of $d+1$ letters, which cannot be written as a product of less than k transpositions (i.e., to each permutation we assign one product). The next proposition gives the analog of the relation (1.5) in the case of a general d.

Proposition 3.1. *The swap matrices $\mathrm{Swap}_{ij}^{(d)}$ satisfy the following degree-reducing relation*

$$\sum_{s\in C_d} s = \sum_{s\in C_{d-1}} s - \sum_{s\in C_{d-2}} s + \cdots + (-1)^{d-1} \sum_{s\in C_1} s + (-1)^d \cdot 1. \tag{3.1}$$

Remark 3.2. We often simplify the notation of sums involving products of swap matrices (e.g., the ones in (3.1)) by summing over (a subset of) the symmetric group and using the fact that every product of swap matrices corresponds to a permutation of the tensor factors of $(\mathbb{C}^d)^{\otimes n}$. On the other hand, every such permutation can be written as a product of swap matrices $\mathrm{Swap}_{ij}^{(d)}$ in a non-redundant way, i.e., relations (1.4) are applied to simplify the expression as much as possible.

Remark 3.3. Note that it is enough to assume that the $d+1$ letters in Equation (3.1) are the numbers $1,\dots,d+1$. In fact, an analogous equation with indices from some other $(d+1)$-subset J of $\{1,\dots,n\}$ can be obtained by conjugating Equation (3.1) with any permutation sending $\{1,\dots,d+1\}$ to J.

Example 3.4. Equation (3.1) for $d=3$ with $(i,j,k,l)=(1,2,3,4)$ is as follows:

$$\begin{gathered}
\mathrm{Swap}_{12}^{(3)}\mathrm{Swap}_{23}^{(3)}\mathrm{Swap}_{34}^{(3)} + \mathrm{Swap}_{12}^{(3)}\mathrm{Swap}_{24}^{(3)}\mathrm{Swap}_{4,3}^{(3)} + \mathrm{Swap}_{13}^{(3)}\mathrm{Swap}_{3,2}^{(3)}\mathrm{Swap}_{24}^{(3)} + \\
\mathrm{Swap}_{13}^{(3)}\mathrm{Swap}_{34}^{(3)}\mathrm{Swap}_{4,2}^{(3)} + \mathrm{Swap}_{14}^{(3)}\mathrm{Swap}_{4,2}^{(3)}\mathrm{Swap}_{23}^{(3)} + \mathrm{Swap}_{14}^{(3)}\mathrm{Swap}_{4,3}^{(3)}\mathrm{Swap}_{3,2}^{(3)} = \\
\mathrm{Swap}_{12}^{(3)}\mathrm{Swap}_{13}^{(3)} + \mathrm{Swap}_{12}^{(3)}\mathrm{Swap}_{14}^{(3)} + \mathrm{Swap}_{12}^{(3)}\mathrm{Swap}_{23}^{(3)} + \mathrm{Swap}_{12}^{(3)}\mathrm{Swap}_{24}^{(3)} + \\
\mathrm{Swap}_{12}^{(3)}\mathrm{Swap}_{34}^{(3)} + \mathrm{Swap}_{13}^{(3)}\mathrm{Swap}_{14}^{(3)} + \mathrm{Swap}_{13}^{(3)}\mathrm{Swap}_{24}^{(3)} + \mathrm{Swap}_{13}^{(3)}\mathrm{Swap}_{34}^{(3)} + \\
\mathrm{Swap}_{14}^{(3)}\mathrm{Swap}_{23}^{(3)} + \mathrm{Swap}_{23}^{(3)}\mathrm{Swap}_{24}^{(3)} + \mathrm{Swap}_{23}^{(3)}\mathrm{Swap}_{34}^{(3)} - \\
\mathrm{Swap}_{12}^{(3)} - \mathrm{Swap}_{13}^{(3)} - \mathrm{Swap}_{14}^{(3)} - \mathrm{Swap}_{23}^{(3)} - \mathrm{Swap}_{24}^{(3)} - \mathrm{Swap}_{34}^{(3)} + 1.
\end{gathered}$$

Remark 3.5. Equation (3.1) can be written in a more condensed form as

$$\sum_{s\in S_{d+1}} \mathrm{sgn}(s)\, s = 0,$$

saying that the *antisymmetrizer* on $d+1$ letters equals zero (see [Pro07, Section 9.3.1]). Here sgn denotes the sign of a permutation and each s is expressed in terms of the swap matrices $\mathrm{Swap}_{ij}^{(d)}$ in a non-redundant way as explained in Remark 3.2.

Next is a preliminary lemma on the way to give an elementary proof of Proposition 3.1.

Lemma 3.6. *Let v be of the form $e_{i_1}\otimes e_{i_2}\otimes\cdots\otimes e_{i_{k+1}}$ where $i_m\in\{1,2,\dots,d\}$ for $m=1,\dots,k+1$ and exactly two of the i_m are the same.*

Then any cycle of length $k+1$ acts on v as a product of $k-1$ transpositions and corresponds to a product of two smaller disjoint cycles (so of length at most k, singletons also count) such that none of them acts on a subset of the tensor factors of v containing two equal factors.

Proof. Let σ be a cycle of length $k+1$ and suppose without loss of generality that 1 is the index i_m that appears twice in v (i.e., v has two copies of e_1). Divide the letters $1,\dots,k+1$ into disjoint tuples B_1,B_2 such that for each i, the indices of the factors of v and $\sigma(v)$ at position $k\in B_i$ are in the same tuple and none of the tuples contains two copies of 1. Then σ is the product of two disjoint cycles, represented by the tuples B_1,B_2. This will also prove that the above set decomposition of $\{1,\dots,k+1\}$ can be done in a unique way.

The algorithm to find the tuples B_i is the following: find a position $j_1\in\{1,\dots,k+1\}$ of one of the two e_1 in v and assign j_1 to B_1. Then consider the factor $e_{i_{m_1}}$ of $\sigma(v)$ at position j_1. If $i_{m_1}=1$, then we add no more elements to B_1 and start the process all over again with the second factor e_1 of v whose position j_2 is assigned to B_2. Otherwise, if $m_1\neq 1$, add j_2 to B_1 and find the position j_2 of $e_{i_{m_1}}$ in v. Consider the factor $e_{i_{m_2}}$ of $\sigma(v)$ at position j_2 and repeat the procedure from before according to whether i_{m_2} equals 1 or not. Proceed until the basis vector $e_{i_{m_r}}$ equals e_1 for some r and we cover all the letters.

Since v only has one index that repeats twice and all the other indices are distinct, this construction gives the desired cyclic decomposition of σ into precisely two shorter cycles. ■

Example 3.7. We provide a concrete example for Lemma 3.6 in the case $d = 5$ and $n = 6$. Let $v = e_1 \otimes e_2 \otimes e_3 \otimes e_4 \otimes e_5 \otimes e_1$ and $\pi = (123456)$ and set

$$\sigma_i = \rho_6^{(5)}(\pi^i), \quad i = 1, \dots, 6.$$

Then

$$\begin{aligned}
\sigma_1(v) &= e_1 \otimes e_1 \otimes e_2 \otimes e_3 \otimes e_4 \otimes e_5 = \rho_6^{(5)}\big((1)(23456)\big)(v),\\
\sigma_2(v) &= e_5 \otimes e_1 \otimes e_1 \otimes e_2 \otimes e_3 \otimes e_4 = \rho_6^{(5)}\big((135)(246)\big)(v),\\
\sigma_3(v) &= e_4 \otimes e_5 \otimes e_1 \otimes e_1 \otimes e_2 \otimes e_3 = \rho_6^{(5)}\big((14)(25)(36)\big)(v),\\
\sigma_4(v) &= e_3 \otimes e_4 \otimes e_5 \otimes e_1 \otimes e_1 \otimes e_2 = \rho_6^{(5)}\big((135)^2(246)^2\big)(v),\\
\sigma_5(v) &= e_2 \otimes e_3 \otimes e_4 \otimes e_5 \otimes e_1 \otimes e_1 = \rho_6^{(5)}\big((12345)(6)\big)(v).
\end{aligned}$$

Proof of Proposition 3.1. First note that it is enough to verify Equation (3.1) on basis vectors v of $\big(\mathbb{C}^d\big)^{\otimes(d+1)}$ of the form $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_{d+1}}$, where $i_m \in \{1, 2, \dots, d\}$ for $m = 1, \dots, d+1$ and exactly two of the i_m are the same. Indeed, if such vectors satisfy (3.1), then the basis vectors with more recurring indices also satisfy (3.1) (introduce new indices for the recurring indices, apply the results for the basis vectors as above and then bring back the old indices). Since (3.1) is invariant under permutation of indices, it is enough to prove that 3.1 holds for basis vectors v of the form

$$v = e_1 \otimes e_2 \otimes e_3 \otimes \cdots \otimes e_d \otimes e_1.$$

Lemma 3.6 shows that, when evaluated on such a basis vector, each term s' in the sum over C_d cancels with a different term s in the sum over C_{d-1} such that none of the disjoint cycles of s acts on a subset of the factors of v with two equal factors. More precisely, if $s' = (1\,2\,3 \cdots d+1) \in C_d$, the images of v under the powers $s', (s')^2, \dots, (s')^d$ are the d permutations with corresponding cyclic structures $(1, d), (2, d-1), \dots, (d, 1)$ such that the two factors e_1 of v are not in the same cycle. Here (i, j) with $i + j = d + 1$ stands for a product of two disjoint cycles, one of length i and one of length j. By permuting the letters of $(s')^j$, we obtain all the elements s in the sum over C_{d-1} corresponding to products with cyclic structure (i, j) for any i, j with $i + j = d + 1$ that separate the two factors e_1 of v.

This means that by applying on v all the terms s' in the sum over C_d, we obtain the actions on v of all the terms s in the sum over C_{d-1} that act on subsets of the factors of v with no equal factors. Hence, the remaining terms in the sum over C_{d-1} are such that one of their disjoint cycles acts on a subset of the factors of v with two equal factors. We then again apply Lemma 3.6 and proceed inductively. ■

4. Identifying the qudit swap algebra $M_n^{\mathrm{Sw}_d}(\mathbb{C})$ as a quotient of the free algebra

Let $\mathbb{C}\langle \mathrm{swap}_{ij} \mid 1 \le i < j \le n\rangle$ be the free $*$-algebra on $\binom{n}{2}$ generators endowed with the involution $*$ that fixes each swap_{ij} and acts as conjugation on $\mathbb{C}$. For $d \in \mathbb{N}$ let $\mathcal{I}_n^{\mathrm{Sw}_d}$

be the its ideal generated by

$$\begin{aligned} \mathrm{swap}_{ij}^2 &= 1, \\ \mathrm{swap}_{ij}\mathrm{swap}_{jk} &= \mathrm{swap}_{ik}\mathrm{swap}_{ij} = \mathrm{swap}_{jk}\mathrm{swap}_{ik}, \\ \mathrm{swap}_{ij}\mathrm{swap}_{kl} &= \mathrm{swap}_{kl}\mathrm{swap}_{ij}, \end{aligned} \tag{4.1}$$

for all distinct indices i, j, k, l, and the relation in (3.1) with the swap matrices $\mathrm{Swap}_{ij}^{(d)}$ replaced by the free variables swap_{ij}. We use the convention that whenever $i > j$, then swap_{ij} is interpreted as swap_{ji}. Denote

$$\mathcal{A}_n^{\mathrm{Sw}_d} := \mathbb{C}\langle \mathrm{swap}_{ij} \mid 1 \le i < j \le n\rangle / \mathcal{I}_n^{\mathrm{Sw}_d}$$

and observe that there is a natural surjective $*$-homomorphism $\rho : \mathbb{C}S_n \to \mathcal{A}_n^{\mathrm{Sw}_d}$ defined by

$$\rho((i,j)) = \mathrm{swap}_{ij} + \mathcal{I}_n^{\mathrm{Sw}_d}. \tag{4.2}$$

We will show that the algebras $\mathcal{A}_n^{\mathrm{Sw}_d}$ and $M_n^{\mathrm{Sw}_d}(\mathbb{C})$ are isomorphic. While this statement follows from [Pro07, Theorem 11.6.1] (i.e., $M_n^{\mathrm{Sw}_d}(\mathbb{C})$ is isomorphic to $\mathbb{C}[S_n]$ modulo the two-sided ideal generated by the antisymmetrizer on $d+1$ letters), we present an elementary self-contained argument.

Proposition 4.1. *The generators of $\mathcal{I}_n^{Sw_d}$ do not all vanish under the irreps of S_n corresponding to partitions λ of n with $\mathrm{ht}(\lambda) > d$.*

Proof. To show that the swap relations (3.1) are not satisfied by any irrep of S_n corresponding to a partition with at least $d+1$ rows, consider an irrep corresponding to a partition $\lambda \vdash n$ of shape $(\lambda_1, \dots, \lambda_k)$ with $k \ge d+1$. We know that any irrep of S_n is spanned by polytabloids

$$e_T = \sum_{\pi \in C_T} \mathrm{sgn}(\pi)\pi\{T\},$$

where T ranges over all tabloids of shape λ, C_T is the set of all permutations that permute the elements only within the columns of T and for each $\pi \in C_T$, $\pi\{T\}$ is the tabloid obtained from T by permuting the entries according to π.

Let T be the standard Young tableaux of shape λ and consider the action of the polynomial

$$g = (-1)^{d-1}\sum_{s\in C_d} s + (-1)^d \sum_{s \in C_{d-1}} s + (-1)^{d-1}\sum_{s\in C_{d-2}} s \pm \dots - \sum_{s \in C_1} s + 1$$

on the polytabloid e_T via

$$s_{ij}\, e_T = e_{(i,j)T}.$$

Choose the indices $i_1, \dots, i_{d+1}$ to be $1, \lambda_1+1, \dots, \lambda_d+1$ respectively and note that the coefficient at T in the resulting polytabloid is

$$(-1)^{d+1}\,|C_d| \cdot \mathrm{sgn}(s \in C_d) + (-1)^d\,|C_{d-1}| \cdot \mathrm{sgn}(s \in C_{d-1}) + \dots + |C_2| \cdot 1 - |C_1| \cdot (-1) + 1.$$

But the latter is strictly positive since, for $k = 1, \dots, d$, the sign of the elements in C_k is $(-1)^k$. This shows that the polynomial g does not vanish under the evaluation $s_{ij} = \rho_\lambda(i\,j)$ for the chosen λ. Thus the swap relations (3.1) are incompatible with any Young Tableaux with more than d rows. ■

Proposition 4.2. *All the irreps of S_n corresponding to a partition of n with at most d rows in its Young tableaux satisfy (3.1).*

Proof. For any irrep of S_n corresponding to a partition $\lambda \vdash n$ with at most d rows it is enough to prove that (3.1) holds when evaluated at basis vectors, i.e., polytabloids

$$e_T = \sum_{\pi \in C_T} \operatorname{sgn}(\pi)\pi\{T\}, \tag{4.3}$$

where T is any tabloid of shape λ. We use the canonical identification between tabloids T and rank one vectors $v \in (\mathbb{C}^d)^{\otimes n}$ of the form $v = e_{i_1} \otimes \cdots \otimes e_{i_k}$ with $i_j \in \{1, \ldots, d\}$. Suppose $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $\lambda_i \geq \lambda_{i+1}$ and let T be a tabloid of shape λ. Define v to be the vector with tensor factors e_k at positions, which are the numbers appearing in the k-th row of T. Now permuting the tensor factors of v is the same as permuting the entries of T. The inverse of this procedure assigns to a rank one vector v the tabloid T with k-th row consisting of the numbers that index the positions of tensor factors e_k in v. Now the proof of Equation (3.1) in Section 3 implies that Equation (3.1) holds when evaluated at each summand in (4.3), from which the claim follows. ∎

Theorem 4.3. *The algebras $\mathcal{A}_n^{\mathrm{Sw}_d}$ and $M_n^{\mathrm{Sw}_d}(\mathbb{C})$ are isomorphic.*

Proof. The algebras $M_n^{\mathrm{Sw}_d}(\mathbb{C})$ and $\mathcal{A}_n^{\mathrm{Sw}_d}$ are both homomorphic images of the semisimple finite-dimensional algebra $\mathbb{C}[S_n]$. Therefore $M_n^{\mathrm{Sw}_d}(\mathbb{C})$ and $\mathcal{A}_n^{\mathrm{Sw}_d}$ and semisimple and finite-dimensional as well. To show that they are isomorphic, we prove that $\mathcal{A}_n^{\mathrm{Sw}_d}$ and $M_n^{\mathrm{Sw}_d}(\mathbb{C})$ have the same block decomposition into simple matrix algebras.

The block decomposition of $M_n^{\mathrm{Sw}_d}(\mathbb{C})$ is described in Theorem 2.2. Recall that $\mathcal{A}_n^{\mathrm{Sw}_d} = \mathbb{C}\langle \mathrm{swap}_{ij} \mid 1 \leq i < j \leq n \rangle / \mathcal{I}_n^{\mathrm{Sw}_d}$, where $\mathcal{I}_n^{\mathrm{Sw}_d}$ is the ideal generated by the relations (4.1) defining the symmetric group S_n and the degree-reducing relation (3.1), which holds precisely on the irreps of S_n indexed by partitions with at most d rows (see Proposition 4.1 and Proposition 4.2). It is now immediate that the algebras $\mathcal{A}_n^{\mathrm{Sw}_d}$ and $M_n^{\mathrm{Sw}_d}(\mathbb{C})$ have the same semisimple decomposition. ∎

5. NPO hierarchy

The identification of the swap algebra $M_n^{\mathrm{Sw}_d}(\mathbb{C})$ as a quotient of the free algebra in Section 4 allows one to view the d-QMC as an example of a noncommutative polynomial optimization (NPO) problem.

Let $\mathcal{F}_n = \mathbb{C}\langle \mathrm{swap}_{ij} \mid 1 \leq i < j \leq n \rangle$ be the $*$-free algebra on $\binom{n}{2}$ generators, and $V_\ell = \{s \in \mathcal{F}_n \colon \deg s \leq \ell\}$ its subspace spanned by the products of at most ℓ swap symbols. Recall the isomorphism between $M_n^{\mathrm{Sw}_d}(\mathbb{C})$ and $\mathcal{A}_n^{\mathrm{Sw}_d} = \mathcal{F}_n / \mathcal{I}_n^{\mathrm{Sw}_d}$ from Theorem 4.3. We can view the Hamiltonian H_G^d from (H_G^d) as an element of $\mathcal{A}_n^{\mathrm{Sw}_d}$, and let

$$h_G = \sum_{(i,j) \in \mathrm{E}(G)} 2w_{ij} \left(I - \mathrm{swap}_{ij}\right)$$

be the corresponding element in $\mathcal{F}_n$. Since the $*$-algebra $M_n^{\mathrm{Sw}_d}(\mathbb{C})$ is finite-dimensional, it is a C*-algebra. Therefore the largest eigenvalue of H_G^d equals

$$\begin{aligned} \alpha_* &= \min\left\{\alpha : \alpha - H_G^d = a^* a \text{ for some } a \in \mathcal{A}_n^{\mathrm{Sw}_d}\right\} \\ &= \min\left\{\alpha : \alpha - h_G = \sum_k s_k^* s_k + q \text{ for some } s_k \in \mathcal{F}_n,\ q \in \mathcal{I}_n^{\mathrm{Sw}_d}\right\}. \end{aligned}$$

For $\ell = 1, \dots, n$ define two sequences,
(5.1)
$$\alpha'_\ell = \min\left\{\alpha : \alpha - h_G = \mathbf{u}_\ell^* \left(A + \sum_m g_m A_m\right)\mathbf{u}_\ell \text{ for some } A_m = A_m^\intercal, \text{ and } A \succeq 0\right\}$$
and
$$\alpha_\ell = \min\left\{\alpha : \alpha - h_G = \sum_k s_k^* s_k + q \text{ for some } s_k \in V_\ell,\ q \in \mathcal{I}_n^{\mathrm{Sw}_d}\right\} \tag{5.2}$$
$$= \min\left\{\alpha : \alpha - h_G \equiv \mathbf{u}_\ell^* A \mathbf{u}_\ell \mod \mathcal{I}_n^{\mathrm{Sw}_d} \text{ for some } A \succeq 0\right\},$$
where $\mathbf{u}_\ell$ is a column of products of at most ℓ swap symbols, and the g_m are the generators of the ideal $\mathcal{I}_n^{\mathrm{Sw}_d}$ as in Section 4. Then $\alpha_\ell \le \alpha'_\ell$ for every ℓ, the sequences $\{\alpha_\ell\}_\ell$ and $\{\alpha'_\ell\}_\ell$ are decreasing, and $\alpha_{n-1} = \alpha'_{n-1} = \alpha_*$. The last equality holds since every permutation in S_n is a product of at most $n-1$ transpositions, and $\mathcal{A}_n^{\mathrm{Sw}_d}$ is a quotient of $\mathbb{C}[S_n]$.

Clearly, (5.1) is a semidefinite program (SDP). The second line in (5.2) is likewise an SDP once the calculation modulo $\mathcal{I}_n^{\mathrm{Sw}_d}$ is resolved (see the paragraph below). We refer to them as the ℓ^{th} relaxations of the d-QMC. Thus we obtained hierarchies of SDPs whose solutions converge to the solution of the d-QMC from below. The hierarchy associated with α'_ℓ is a very special case of the analog of the Lasserre hierarchy [Lse01] for NPO that is based on a noncommutative Positivstellensatz [HM04], and whose dual is the Navascués-Pironio-Acín hierarchy [NPA08, PNA10] in quantum physics.

While the expression (5.1) is readily an SDP, it involves more unknowns than (5.2) (i.e., in addition to unknowns α and $A \succeq 0$, it also involves several unknown symmetric A_m). Thus, it is preferable to work with (5.2). To prepare the linear constraints in the SDP (5.2) that arise from $\alpha - h_G \equiv \mathbf{u}_\ell^* A \mathbf{u}_\ell \mod \mathcal{I}_n^{\mathrm{Sw}_d}$ (note that the right-hand side involves products of at most 2ℓ swap symbols), one needs to identify a subset $\mathcal{B}_{2\ell}^d$ in $V_{2\ell}$ that maps to a basis under the quotient map $q : V_{2\ell} \to (V_{2\ell} + \mathcal{I}_n^{\mathrm{Sw}_d})/\mathcal{I}_n^{\mathrm{Sw}_d}$. To do this, one can start with a basis of $V_{2\ell}$, reduce it modulo $\mathcal{I}_n^{\mathrm{Sw}_d}$ via a noncommutative Gröbner basis algorithm [Mor86], and then identify a basis $\mathcal{B}_{2\ell}^d$ in the resulting set. Alternatively, one can obtain a concrete instance of $\mathcal{B}_{2\ell}^d$ as follows. In [Pro21], a permutation $\pi \in S_n$ is called $(d+1)$-**good** if there is no increasing sequence $j_0 < \cdots < j_d$ such that $\pi(j_0) > \cdots > \pi(j_d)$. By [Pro21, Theorem 8], $(d+1)$-good permutations form a basis of $M_n^{\mathrm{Sw}_d}(\mathbb{C})$. For $\mathcal{B}_{2\ell}^d$ one can thus choose the set of all $(d+1)$-good permutations that are products of at most 2ℓ transpositions.

When ℓ is large, the size of the SDP for α_ℓ (i.e., the number of variables, linear constraints, and the size of the semidefinite constrain) is typically too large for available SDP solvers. In practice, one thus often has to settle for computing only the first two relaxations of α_*, namely α_1 and α_2. To solve these two SDPs, the sets $\mathcal{B}_2^d$ and $\mathcal{B}_4^d$ are required in view of the preceding paragraph. For $d = 2$, these are given in [BCEHK24, Subsection 4.3.2 and Appendix B.2]. For $d \ge 5$, one can take $\mathcal{B}_4^d$ (resp. $\mathcal{B}_2^d$) consisting of all permutations that are products of at most 4 (resp. 2) transpositions; see Appendix A. For $d \in \{3, 4\}$, the bases are presented in Appendices B.1, B.4 and B.6.

Example 5.1. We computed the first two relaxations of (5.2) in the case $d = 3$ for all 853 connected graphs on $n = 7$ vertices. The list of graphs was generated using Nauty [MP14]. To construct the SDP forms of (5.2) we used noncommutative Gröbner bases computed with Magma [BCP97]; alternately, the results of Appendices B.1 and

B.4 could be employed. The produced SDPs were solved on a Macbook Air laptop using Mathematica[2]. The second relaxation was (up to numerical precision) exact on all seven vertex graphs. On the other hand, the first relaxation performed very poorly. The reason is that in low degrees (so degree ≤ 2 when working with the first relaxation) the nontrivial relation (3.1) defining the 3-swap algebra does not enter computations. One is thus essentially only optimizing over the corresponding group algebra, where the solution is trivially found; cf. Subsection 6.1.

Thus the 3-QMC provides a large class of examples where the second NPO relaxation clearly outperforms the first one.

6. Quantum max d-cut and irreps

The decomposition of the d-swap algebra $M_n^{\mathrm{Sw}_d}(\mathbb{C})$ described in Section 2.2 is a valuable tool for calculating the eigenvalues of the qudit Quantum Max Cut Hamiltonian of a complete graph on n vertices. Recall from (1.3) that given a graph G, the d-QMC irrep Hamiltonian H_G^d is defined as

$$H_G^d = \rho_n^{(d)}\left(\sum_{(i,j)\in \mathrm{E}(G)} 2w_{ij}\left(I-(i\ j)\right)\right) = \sum_{(i,j)\in \mathrm{E}(G)} 2w_{ij}\left(I-\mathrm{Swap}_{ij}^{(d)}\right).$$

Here the $\mathrm{Swap}_{ij}^{(d)}$ denote the qudit swap matrices in $M_n^{\mathrm{Sw}_d}(\mathbb{C})$.

Definition 6.1. Let G be a graph on n vertices with edge set $\mathrm{E}(G)$ and edge weights w_{ij}. Let $\lambda \vdash n$ be a partition labelling an irrep of S_n. The **QMC irrep Hamiltonian** H_G^λ is defined as

$$H_G^\lambda = \rho_\lambda\left(\sum_{(i,j)\in \mathrm{E}(G)} 2w_{ij}\left(I-(i\ j)\right)\right).$$

The following is a straightforward corollary of Theorem 2.2.

Corollary 6.2. *The spectrum of the d-QMC Hamiltonian of a graph G is the union of the spectra of all the Hamiltonians corresponding to the irreps of S_n with at most d rows. That is,*

$$\mathrm{eigs}(H_G^d) = \bigcup_{\substack{\lambda\vdash n\\ \mathrm{ht}(\lambda)\leq d}} \mathrm{eigs}(H_G^\lambda),$$

and, in particular,

$$\mathrm{eig}_{\max}(H_G^d) = \max_{\substack{\lambda\vdash n\\ \mathrm{ht}(\lambda)\leq d}} \left(\mathrm{eig}_{\max}(H_G^\lambda)\right).$$

6.1. Exact solution for sufficiently large d.

We record the largest eigenvalue of the Hamiltonian

$$H_G^d = \sum_{(i,j)\in \mathrm{E}(G)} 2w_{ij}\left(I-\mathrm{Swap}_{ij}^{(d)}\right)$$

if $d \geq n = |\mathrm{V}(G)|$ and $w_{ij} \geq 0$ for all $(i,j) \in \mathrm{E}(G)$.

Proposition 6.3. *If all the edge weights in G are nonnegative and $d \geq n$, the largest eigenvalue of H_G^d is $4\sum_{i,j} w_{ij}$.*

[2] https://www.wolfram.com/mathematica

Proof. Clearly,

$$\|H_G^d\| \le \sum_{(i,j)\in \mathrm{E}(G)} 2w_{ij} \left\| I - \mathrm{Swap}_{ij}^{(d)} \right\| = 4\sum_{i,j} w_{ij},$$

so the largest eigenvalue of H_G^d is at most $4\sum_{i,j} w_{ij}$. If $d \ge n$, then

$$v = \sum_{\pi\in S_n} \mathrm{sgn}(\pi) e_{\pi(1)} \otimes \cdots \otimes e_{\pi(n)}$$

satisfies $\mathrm{Swap}_{ij}^{(d)} v = -v$ for all $i \ne j$. Therefore $H_G^d v = \left(4\sum_{i,j} w_{ij}\right) v$. ∎

Let us end this short subsection with a comment on the case $d = n - 1$. While $M_n^{\mathrm{Sw}_n}(\mathbb{C}) \cong \mathbb{C}[S_n]$, the swap algebra $M_n^{\mathrm{Sw}_{n-1}}(\mathbb{C})$ is isomorphic to the direct sum of all the irreps of S_n apart from the one-dimensional sign representation of S_n. The latter is, as a sub-representation of $\mathbb{C}[S_n]$, spanned by $a = \frac{1}{n!}\sum_{\pi\in S_n} \mathrm{sgn}(\pi)\pi$, the antisymmetrizer in $\mathbb{C}[S_n]$. Thus $M_n^{\mathrm{Sw}_{n-1}}(\mathbb{C})$ is, as a C*-algebra, isomorphic to the orthogonal complement of a in $\mathbb{C}[S_n]$. Under this identification, the Hamiltonian H_G^{n-1} corresponds to

$$\sum_{(i,j)\in \mathrm{E}(G)} 2w_{ij}\big(\mathrm{id} - (i\ j)\big) - 4\left(\sum_{(i,j)\in \mathrm{E}(G)} w_{ij}\right) a \in \mathbb{C}[S_n] \tag{6.1}$$

because the projection of $\mathrm{id} - (i\ j)$ onto the span of a equals $2a$. While (6.1) lacks the sparsity (2-locality) of H_G^{n-1}, it can at least be viewed as an operator on a slightly smaller space of dimension $n! < (n-1)^n$ via the left regular representation of S_n. We speculate that $M_n^{\mathrm{Sw}_{n-1}}(\mathbb{C})$ differing from $\mathbb{C}[S_n]$ only for the (very simple) sign representation might offer further insight into the $(n-1)$-QMC problem, which is currently beyond reach.

6.2. **Exact solutions for clique Hamiltonians with uniform edge weights.** We now present the main steps in the computation of the spectrum of the d-QMC Hamiltonian of a complete graph with uniform edge weights. For the rest of this section we assume all edge weights $w_{ij} = 1$.

The clique is the easiest graph for tackling the d-QMC problem since the isotypic components of the d-QMC Hamiltonian are scalar matrices in this case.

Lemma 6.4. *Let $\lambda \vdash n$ be a partition. Then*

$$H_{K_n}^{\lambda} = \eta_\lambda I, \tag{6.2}$$

where η_λ is a scalar depending only on the irrep λ and I is the identity matrix of the appropriate dimension.

Proof. Follows by [BCEHK24, Lemma 2.11]. ∎

For any partition $\lambda \vdash n$, the dimension of the irrep ρ_λ of S_n is the value of the corresponding character $\chi_\lambda : S_n \to \mathbb{C}$ at the identity element $e \in S_n$. So

$$\chi_\lambda(\pi) = \mathrm{Tr}(\rho_\lambda(\pi)), \qquad \pi \in S_n,$$

and, in particular,

$$\chi_\lambda(e) = \mathrm{Tr}(\rho_\lambda(e))$$

is the dimension of the irrep ρ_λ of S_n. From Lemma 6.4 if follows that the eigenvalue η_λ can be expressed through the values of the character χ_λ at the identity e and at any transposition $(i\,j)$.

Lemma 6.5. *For any $\lambda \vdash n$ let χ_λ be the character corresponding to ρ_λ and let η_λ be as in Lemma 6.4. Then*

$$\eta_\lambda = 2\binom{n}{2}\left(1 - \frac{\chi_\lambda\big((i\ j)\big)}{\chi_\lambda(e)}\right). \tag{6.3}$$

Proof. For $\lambda \vdash n$, the constant η_λ can be explicitly computed by taking the trace on both sides of (6.2). Indeed, since

$$H^\lambda_{K_n} = \rho_\lambda\left(\sum_{(i,j)\in \mathrm{E}(G)} 2\left(I - (i\ j)\right)\right),$$

we get, by taking the trace, that

$$\mathrm{Tr}\big[H^\lambda_{K_n}\big] = \sum_{(i,j)\in \mathrm{E}(G)} 2\left[\chi_\lambda(e) - \chi_\lambda\big((i\ j)\big)\right] = 2\binom{n}{2}\left[\chi_\lambda(e) - \chi_\lambda\big((i\ j)\big)\right].$$

On the other hand,

$$\mathrm{Tr}\big[H^\lambda_{K_n}\big] = \eta_\lambda\, \chi_\lambda(e),$$

so that

$$\eta_\lambda = 2\binom{n}{2}\left(1 - \frac{\chi_\lambda\big((i\ j)\big)}{\chi_\lambda(e)}\right). \qquad \blacksquare$$

Example 6.6. For a two-row partition $\lambda = (n-k, k)$, it was computed in [BCEHK24, Lemma 2.12] that

$$\eta_\lambda = 2k(n+1) - 2k^2.$$

We now compute the eigenvalue η_λ for any partition λ using a formula by Frobenius [Fro00]. For a more direct approach, where we explicitly compute the value of χ_λ at a transposition using the well-known hook-length formula, see Appendix C.

Proposition 6.7. *Let η_λ be as in Lemma 6.4. For any $\lambda \vdash n$ with rows $\lambda_1 \geq \cdots \geq \lambda_d$,*

$$\eta_\lambda = n^2 + \frac{d(d-1)(2d-1)}{6} - \sum_{k=1}^{d}\left(\lambda_k - (k-1)\right)^2. \tag{6.4}$$

Proof. Let $\lambda \vdash n$ be a partition with rows $\lambda_1 \geq \cdots \geq \lambda_d \geq 1$. Recall that the conjugate partition λ' of λ is the partition of n, whose k-th row is the k-th column of λ. It follows from [Fro00, p. 534] (or [Lsa08, Theorem 4]) that for any transposition $(i\,j)$,

$$\binom{n}{2}\frac{\chi_\lambda\big((i\ j)\big)}{\chi_\lambda(e)} = \sum_{k=1}^{d}\left[\binom{\lambda_k}{2} - \binom{\lambda'_k}{2}\right].$$

Moreover, by [Sta99, Proposition 1.8.3] we have

$$\sum_{k=1}^{d}\binom{\lambda'_k}{2} = \sum_{i=1}^{d}(k-1)\lambda_k.$$

Hence,

$$\begin{aligned}\eta_\lambda &= 2\binom{n}{2}\left(1 - \frac{\chi_\lambda\big((i\ j)\big)}{\chi_\lambda(e)}\right)\\ &= 2\binom{n}{2} - 2\sum_{k=1}^{d}\left[\binom{\lambda_k}{2} - \binom{\lambda'_k}{2}\right]\end{aligned}$$

$$= 2\binom{n}{2} - 2\sum_{k=1}^{d}\left[\binom{\lambda_k}{2} - (k-1)\lambda_k\right]$$
$$= n(n-1) + \sum_{k=1}^{d}\lambda_k - \sum_{k=1}^{d}\left(\lambda_k^2 - 2(k-1)\lambda_k\right)$$
$$= n^2 - \sum_{k=1}^{d}\left(\lambda_k^2 - 2(k-1)\lambda_k\right)$$
$$= n^2 + \frac{d(d-1)(2d-1)}{6} - \sum_{k=1}^{d}\left(\lambda_k - (k-1)\right)^2. \qquad \blacksquare$$

Using Proposition 6.7, one can deduce the solution to the d-QMC problem for a clique, i.e., the maximal η_λ, where $\lambda \vdash n$ ranges over all partitions with at most d rows. Moreover, the form (6.4) of η_λ eases the computation of the precise partition $\lambda \vdash n$ at which the maximum is obtained.

Corollary 6.8. *The maximum value of η_λ among all partitions $\lambda \vdash n$ with $\mathrm{ht}(\lambda) \le d$ is obtained at*

$$\lambda = \Big(\underbrace{1+\frac{n-r}{d},\dots,1+\frac{n-r}{d}}_{r},\ \underbrace{\frac{n-r}{d},\dots,\frac{n-r}{d}}_{d-r}\Big) \tag{6.5}$$

for $n \equiv r \pmod d$. Moreover, the solution to the d-QMC problem for an n-clique is

$$n^2 + (d-1)n + r^2 - r(d+1) - \frac{n^2-r^2}{d}. \tag{6.6}$$

Proof. The statement for $d = n$ is routine (or see Proposition 6.3). We thus assume $d < n$.

First, for each partition $\lambda \vdash n$ with $e = \mathrm{ht}(\lambda) < d$ we find a partition $\tilde\lambda \vdash n$ with $\mathrm{ht}(\tilde\lambda) = e+1 \le d$ such that $\eta_{\tilde\lambda} > \eta_\lambda$. Let e' be the largest index for which $\lambda_{e'} > 1$ (e' exists since $d < n$). Then construct $\tilde\lambda \vdash n$ with $\mathrm{ht}(\tilde\lambda) = e+1$ as follows:

$$\tilde\lambda_j = \begin{cases} \lambda_j & 1 \le j \le e,\ j \ne e' \\ \lambda_j - 1 & j = e' \\ 1 & j = e+1. \end{cases}$$

Now

$$\eta_\lambda - \eta_{\tilde\lambda} = n^2 + \frac{e(e-1)(2e-1)}{6} - \sum_{k=1}^{e}\big(\lambda_k - (k-1)\big)^2$$
$$- \Big(n^2 + \frac{(e+1)e(2e+1)}{6} - \sum_{k=1}^{e+1}\big(\tilde\lambda_k - (k-1)\big)^2\Big)$$
$$= -e^2 - (\lambda_{e'} - (e'-1))^2 + (\lambda_{e'} - 1 - (e'-1))^2 + (1-e)^2$$
$$= -2(\lambda_{e'} + e - e') < 0,$$

as desired.

Thus the solution to the d-QMC problem is attained at $\lambda \vdash n$ with $\mathrm{ht}(\lambda) = d$. Since n, d are fixed, maximizing η_λ is by Proposition 6.7 equivalent to minimizing

$$f(\lambda) := \sum_{k=1}^{d} \big(\lambda_k - (k-1)\big)^2 \tag{6.7}$$

over partitions $\lambda \vdash n$ with $\mathrm{ht}(\lambda) = d$. That is, $\sum_{k=1}^{d} \lambda_k = n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq 1$.

We claim that any minimizer $\lambda^\star$ of (6.7) has at most one jump, i.e., $\lambda_1^\star - \lambda_d^\star \leq 1$. Assume otherwise. Then there are $d \geq k > \ell \geq 2$ such that

$$\lambda_d^\star \leq \cdots \leq \lambda_k^\star < \lambda_{k-1}^\star \leq \cdots \leq \lambda_\ell^\star < \lambda_{\ell-1}^\star \leq \cdots \leq \lambda_1.$$

We now replace λ_k^* with $\lambda_k^* + 1$ and $\lambda_{\ell-1}^*$ with $\lambda_{\ell-1}^* - 1$ to obtain a new partition $\lambda^\dagger \vdash n$ with $\mathrm{ht}(\lambda^\dagger) \leq d$. Then

$$\begin{aligned} f(\lambda^*) - f(\lambda^\dagger) &= \big(\lambda_k - (k-1)\big)^2 + \big(\lambda_{\ell-1} - (\ell-2)\big)^2 \\ &\quad - \big(\lambda_k + 1 - (k-1)\big)^2 - \big(\lambda_{\ell-1} - 1 - (\ell-2)\big)^2 \\ &= 2(k-\ell) + 2(\lambda_{\ell-1} - \lambda_k) > 0, \end{aligned}$$

contradicting the minimality of $\lambda^\star$. Since a minimizer λ of (6.7) satisfies $\lambda_1 - \lambda_d \leq 1$, (6.5) follows.

It is routine to check that the solution to the d-QMC problem for an n-clique, obtained by plugging (6.5) into the formula (6.4) for η_λ, is in fact (6.6). ∎

Remark 6.9. The solution (6.6) to the d-QMC problem for an n-clique is indeed an integer, since

$$\frac{n^2 - r^2}{d} = \frac{(n-r)(n+r)}{d}$$

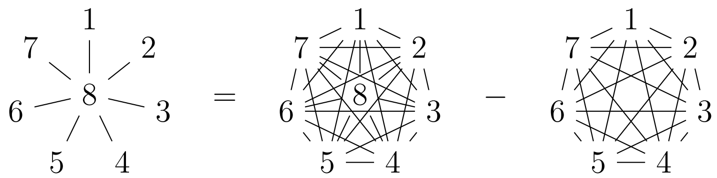
and $n - r$ is divisible by d (as $n \equiv r \pmod d$).

6.3. **Graph clique decomposition.** In this section we use an algorithm from [BCEHK24, Section 6], called *graph clique decomposition*, to solve the d-QMC problem for a larger family of graphs, namely star graphs and a large class of complete bipartite graphs. The clique decomposition is an expression of the d-QMC Hamiltonian of a given graph as an alternating sum of d-QMC Hamiltonians of cliques and simple graphs.

6.3.1. *Exact solutions for star graphs.* Let $n \geq 2$, and consider the star graph $\bigstar_n$ on n-vertices and observe that if we label the vertices of $\bigstar_n$ so that n corresponds to the central vertex, then

$$\bigstar_n = K_n - K_{n-1}. \tag{6.8}$$

E.g., for $n = 8$ we have



Here, we view K_{n-1} as a graph on n vertices, in which the vertex n is disconnected from the rest. This is the clique decomposition of $\bigstar_n$, which together with the Young branching rule [Sag01, § 2] facilitates the computation of the eigenvalues of $H^d_{\bigstar_n}$ significantly. The spectrum of $H^d_{\bigstar_n}$ in the case $d = 2$ was computed in [BCEHK24]. Note that $H^{(n)}_{\bigstar_n} = 0$ for the 1-row partition $(n) \vdash n$.

Example 6.10 ([BCEHK24, Lemma 6.1])**.** Let $n \geq 2$ and $\lambda = (\lambda_1, \lambda_2)$. If $\lambda_1 > \lambda_2$ then $H^\lambda_{\bigstar_n}$ has two eigenvalues

$$e_1 = 2(n - \lambda_1), \qquad e_2 = 2(n - \lambda_2 + 1).$$

If $\lambda_1 = \lambda_2$ then $H^\lambda_{\bigstar_n}$ has only one eigenvalue $e_1 = 2(n - \lambda_2 + 1) = n + 2$. The solution to the 2-QMC problem for $\bigstar_n$ is $2n$, attained at the partition $\lambda = (n - 1, 1)$.

We extend this result by computing the eigenvalues of $H^d_{\bigstar_n}$.

Theorem 6.11. *If $\lambda = (\lambda_1, \dots, \lambda_e) \vdash n$ has $e \leq d$ rows $\lambda_1 \geq \cdots \geq \lambda_e \geq 1$, then the eigenvalues of the d-QMC irrep Hamiltonian $H^\lambda_{\bigstar_n}$ form a subset of*

$$\{2(n - \lambda_1),\, 2(n - \lambda_2 + 1),\, \dots,\, 2(n - \lambda_e + e - 1)\}$$

containing the value $\eta_\star = 2(n - \lambda_e + e - 1)$. Hence, the solution to the d-QMC problem for $\bigstar_n$ is $2(n + d - 2)$, attained at any partition with $\lambda_d = 1$.

Proof. From (6.8) we deduce that

$$H^\lambda_{\bigstar_n} = H^\lambda_{K_n} - H^\lambda_{K_{n-1}}$$

for any partition λ. By Lemma 6.4, the first term $H^\lambda_{K_n}$ is a scalar matrix. The eigenvalues of $H^\lambda_{K_{n-1}}$ can be computed using the Young branching rule [Sag01, § 2]. It states that the restriction of any irrep, say labeled by the partition λ, of S_n to the subgroup S_{n-1} decomposes as a direct sum of all the irreps of S_{n-1} which can be obtained from λ by removing one box.

Hence, the eigenvalues of $H^\lambda_{\bigstar_n}$ are obtained by subtracting from η_λ (which is the single eigenvalue of $H^\lambda_{K_n}$) each of the (at most e) eigenvalues of $H^\lambda_{K_{n-1}}$. Precisely, we use the following procedure: Let the index j run from e to 1 and let η_λ be the eigenvalue of $H^\lambda_{K_n}$ and η_{μ_j} the eigenvalue of $H^{\mu_j}_{K_{n-1}}$, where μ_j is obtained from λ by removing one box from the j-th row. Now let

$$\eta_*(j) = \eta_\lambda - \eta_{\mu_j} = 2(n - \lambda_j + j - 1).$$

As noted, $\eta_* = \eta_*(e)$ is an eigenvalue of $H^\lambda_{\bigstar_n}$ if $\lambda_e \geq 1$, since in this case one can remove a box from the last row of the Young diagram of λ to obtain a valid Young diagram of a partition of $n-1$. Next, consider $j = e-1$. If $\lambda_{e-1} > \lambda_e$ (hence, $\lambda_{e-1} \geq 2$), then $\eta_*(e-1)$ is an eigenvalue of $H^\lambda_{\bigstar_n}$, because one box can be removed from λ_{e-1} to obtain a valid partition of $n - 1$; otherwise, if $\lambda_{e-1} = \lambda_e$, proceed to $j = e - 2$ and so on.

It is now immediate that the largest eigenvalue of $H^d_{\bigstar_n}$ (for $d \leq n$) is $2(n + d - 2)$, which is obtained by plugging $j = d$ and $\lambda_d = 1$ into the expression for $\eta_\star$. ■

To give a more precise description of the spectrum of $H^\lambda_{\bigstar_n}$ for λ with $\mathrm{ht}(\lambda) \leq d$, it is easier to look at that of $nI - \frac{1}{2}H^\lambda_{\bigstar_n}$: its eigenvalues are obtained from the strictly decreasing sequence

$$\lambda_1,\, \lambda_2 - 1,\, \dots,\, \lambda_d - (d-1)$$

by keeping only the smallest element of any subsequence of consecutive values, and then removing $-(d-1)$ if necessary. Indeed, the subsequences of consecutive values correspond to rows in λ with equal length (thus when restricting the irrep to S_{n-1}, a box can be removed only from the lowest such row), while removing $-(d-1)$ corresponds to the possibility that λ has less than d rows.

As an example we now explicitly present the spectrum of $H^3_{\star_n}$.

Example 6.12. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ be a partition of n with three rows. The d-QMC irrep Hamiltonian $H^\lambda_{\star_n}$ has at most three distinct eigenvalues, namely

(1) if $\lambda_1 > \lambda_2 > \lambda_3$ then it has three eigenvalues

$$e_1 = 2(n - \lambda_3 + 2), \quad e_2 = 2(n - \lambda_2 + 1), \quad e_3 = 2(n - \lambda_1),$$

(2) if $\lambda_1 = \lambda_2 > \lambda_3$ then it has two eigenvalues

$$e_1 = 2(n - \lambda_3 + 2), \quad e_2 = 2(n - \lambda_2 + 1),$$

(3) if $\lambda_1 > \lambda_2 = \lambda_3$ then it has two eigenvalues

$$e_1 = 2(n - \lambda_3 + 2), \quad e_2 = 2(n - \lambda_1),$$

(4) if $\lambda_1 = \lambda_2 = \lambda_3$ then it has one eigenvalue

$$e_1 = 2(n - \lambda_3 + 2).$$

The solution to the 3-QMC problem for $\star_n$ is $2(n+1)$, attained at partition of the form $\lambda = (\lambda_1, \lambda_2, 1)$.

Corollary 6.13. *Let $n \geq 2$. If λ, μ are partitions of n with distinct parts, then*

$$\operatorname{spec}(H^\lambda_{\star_n}) = \operatorname{spec}(H^\mu_{\star_n}) \quad \Longleftrightarrow \quad \lambda = \mu.$$

Remark 6.14. The assumption about distinct parts in Corollary 6.13 is necessary. Indeed, $\lambda = (4,2,2,2,2)$ and $\mu = (5,5,1,1)$ satisfy $\operatorname{spec}(H^\lambda_{\star_{12}}) = \{16, 28\} = \operatorname{spec}(H^\mu_{\star_{12}})$. With a bit more effort, one also can find distinct partitions with equal height such that $H^\lambda_{\star_n}$ and $H^\mu_{\star_n}$ have the same eigenvalues:

$$\lambda = (8,5,5,5,5,2,2),$$
$$\mu = (9,9,4,4,2,2,2)$$

satisfy $\operatorname{spec}(H^\lambda_{\star_{32}}) = \{24, 31, 36\} = \operatorname{spec}(H^\mu_{\star_{32}})$.

6.3.2. *Exact solutions for complete bipartite graphs.* The star graph $\star_n$ from the previous section is a special example of a complete bipartite graph. We now describe how the solution to the d-QMC problem can be obtained for a large class of complete bipartite graphs $K_{n-k,k}$ with $k \leq n/2$; by this we mean a graph whose vertices are separated into two subsets N_1 and N_2 of size $n-k$ and k respectively, each with no internal connections and such that every vertex in N_1 is connected to every vertex in N_2. The complement of $K_{n-k,k}$ consists of two cliques K_{n-k} and K_k (cf. Equation (6.8)), hence

$$K_{n-k,k} = K_n - (K_{n-k} \oplus K_k).$$

This gives a formula for the d-QMC Hamiltonian $H^d_{K_{n-k,k}}$,

$$H^d_{K_{n-k,k}} = H^d_{K_n} - (H^d_{K_{n-k}} + H^d_{K_k}),$$

and its blocks, i.e., the d-QMC irrep Hamiltonians,

$$H^\lambda_{K_{n-k,k}} = H^\lambda_{K_n} - (H^\lambda_{K_{n-k}} + H^\lambda_{K_k}),$$

where $\lambda \vdash n$ is a partition of n with at most d rows. Note that the matrices $H^\lambda_{K_{n-k}}$ and $H^\lambda_{K_k}$, and hence also $H^d_{K_{n-k}}$ and $H^d_{K_k}$, commute. So the eigenvalues of $H^\lambda_{K_{n-k,k}}$ are obtained by subtracting from the single eigenvalue of $H^{\lambda}_{K_n}$ the elements of the Minkowski sum of the sets of eigenvalues of $H^{\lambda}_{K_{n-k}}$ and $H^{\lambda}_{K_k}$.

To compute the eigenvalues of $H^{\lambda}_{K_{n-k}}$ and $H^{\lambda}_{K_k}$ one needs to understand the action of the irrep λ of S_n on S_{n-k} and S_k. Similarly to the case of the star graph, the Young branching rule is invoked, but then inductively applied multiple times. In this way we write $H^{\lambda}_{K_{n-k}}$ as a direct sum of certain irreps of S_{n-k} and similarly for $H^{\lambda}_{K_k}$. More precisely,

$$H^{\lambda}_{K_{n-k,k}} = H^{\lambda}_{K_n} - \left(\bigoplus_{\mu} (H^{\mu}_{K_{n-k}})^{n_\mu} + \bigoplus_{\nu} (H^{\nu}_{K_k})^{n_\nu} \right), \tag{6.9}$$

where the first big direct sum is over partitions $\mu \vdash n-k$, obtained from λ by removing k boxes and the second big direct sum is over partitions $\nu \vdash k$, obtained from λ by removing $n-k$ boxes. The power n_μ (resp. n_ν) is the number of ways in which the partition μ (resp. ν) can be obtained from λ by removing k (resp. $n-k$) boxes. Observe that one of the direct summands in the second big direct sum in (6.9) is zero if $\lambda_1 \geq k$.

Definition 6.15. Let $\lambda = (\lambda_1, \dots, \lambda_f) \vdash n$ and let $1 \leq e < f$. Then $\xi = (\lambda_{e+1}, \dots, \lambda_f)$ is a **tail** of λ. The partition ξ is **balanced** if $\lambda_{e+1} - \lambda_f \leq 1$.

Theorem 6.16. *Let $2k \leq n$ and $(k-1)d < n$.*

(1) *If $d \geq 1 + \lfloor \sqrt{k} \rfloor$, then the solution to the d-QMC problem for $K_{n-k,k}$ is attained at any partition λ of height d with $\lambda_1 \geq k$ whose tail is a balanced partition of k of smallest possible height (which is equal to $\lfloor \sqrt{k} \rfloor$ when $k \geq 4$). The value of the solution is*

$$2(n+d)k - \left(\lfloor \sqrt{k} \rfloor + 1\right)(k+r)\left(1 + \frac{k-r}{\lfloor \sqrt{k} \rfloor}\right), \tag{6.10}$$

where $r \equiv k \pmod{\lfloor \sqrt{k} \rfloor}$.

(2) *If $d < 1 + \lfloor \sqrt{k} \rfloor$, then the solution to the d-QMC problem for $K_{n-k,k}$ is attained at the partition $\lambda \vdash n$ with d rows and $\lambda_1 = n-k$ such that $(\lambda_2, \dots, \lambda_d)$ is a balanced partition of k. The value of the solution is*

$$2(n+d)k - d(k+r)\left(1 + \frac{k-r}{d-1}\right) \tag{6.11}$$

where $r \equiv k \pmod{d-1}$.

Proof. To find the maximal eigenvalue of the d-QMC Hamiltonian $H^d_{K_{n-k,k}}$, by the above calculation (6.9), we are maximizing

$$\eta_\lambda - (\eta_\mu + \eta_\nu), \tag{6.12}$$

where $\lambda \vdash n$ has $f \leq d$ rows, the partition $\mu \vdash n-k$ with e rows is obtained from λ by removing k boxes and the partition $\nu \vdash k$ with g rows is obtained from λ by removing $n-k$ boxes.

Since $(k-1)d < n$, we have $\lambda_1 \geq k$, so the second big direct sum in (6.9) has the term $(H^{\nu}_{K_k})^{n_\nu}$ with $\nu = (k) \vdash k$. Hence $\eta_{(k)} = 0$ is the minimal value among the η_ν. Thus (6.12) simplifies to

$$\eta_\lambda - \eta_\mu. \tag{6.13}$$

The proof of Theorem 6.16 is divided into the following six lemmas. We will show that if $d \geq 1 + \lfloor \sqrt{k} \rfloor$, then the maximum is attained at any partition λ of height d with $\lambda_1 \geq k$

whose tail is a balanced partition of k of smallest possible height $d-e \geq \lfloor\sqrt{k}\rfloor$ (and μ is obtained from λ by removing this tail). The aforementioned lemmas derive this conclusion by carefully analyzing how (6.13) changes when a single box in λ or μ is moved (cf. Figure 6.3.2).

Note that for any $k \geq 2$, the balanced partition ξ of k with $\lfloor\sqrt{k}\rfloor$ rows indeed exists. Suppose dividing k with $\lfloor\sqrt{k}\rfloor$ gives quotient s and remainder r. Then define ξ by $\xi_i = s+1$ for $i = 1, \dots, r$ and $\xi_i = s$ for $i = r+1, \dots, \lfloor\sqrt{k}\rfloor$. The constructed ξ is clearly a balanced partition of k with $\lfloor\sqrt{k}\rfloor$ rows.

Further, given n, k, d, a sufficient condition for the existence of a partition $\lambda \vdash n$ with $\lambda_1 \geq k$ and a balanced tail of height $\lfloor\sqrt{k}\rfloor$ is the following:

$$2k + (d - \lfloor\sqrt{k}\rfloor - 1)\left\lceil \frac{k}{\lfloor\sqrt{k}\rfloor} \right\rceil \leq n.$$

The condition $(k-1)d < n$ is a stricter requirement than the above whenever $k \geq 4$. Hence, to maximize (6.13) when $k \geq 4$, the height $d-e$ must in fact be equal to $\lfloor\sqrt{k}\rfloor$. The formula (6.10) is then obtained by plugging the maximizing λ into (6.13) using the expression (6.4).

For $k = 2$, we need to check that the maximal values of (6.13) when the smallest possible height of the tail is 1 or 2 coincide and that they coincide with (6.10). Plugging in $k = 2$ into (6.10) yields $4(n+d-3)$. If $e = d-1$, then

$$\begin{aligned}\eta_\lambda - \eta_\mu &= n^2 + \frac{(d-1)(d-2)(2d-3)}{6} - (n-2)^2 - \frac{(d-2)(d-3)(2d-5)}{6} - (d-4)^2 \\ &= 4(n+d-3).\end{aligned}$$

If $e = d-2$, then

$$\begin{aligned}\eta_\lambda - \eta_\mu = n^2 &+ \frac{d(d-1)(2d-1)}{6} - (n-2)^2 - \frac{(d-2)(d-3)(2d-5)}{6} \\ &- (d-3)^2 - (d-2)^2 \\ &= 4(n+d-3).\end{aligned}$$

It remains to check that when $k = 3$, the maximal values of (6.13) when the smallest possible height of the tail is $1, 2$ or 3 coincide and that they coincide with (6.10). Plugging in $k = 3$ into (6.10) yields $6(n+d-4)$. If $e = d-1$, then the tail is $(3) \vdash 3$ and the value of (6.21) is

$$n^2 - (n-3)^2 - \lambda_d^2 + 2(d-1)\lambda_d = 6(n+d-4). \tag{6.14}$$

If $e = d-2$, the tail is $(2,1) \vdash 3$ and the value of (6.21) is

$$\eta_\lambda - \eta_\mu = n^2 - (n-3)^2 - \lambda_{d-1}^2 + 2(d-2)\lambda_{d-1} - \lambda_d^2 + 2(d-1)\lambda_d = 6(n+d-4). \tag{6.15}$$

Finally, if $e = d-3$, then the tail is $(1,1,1) \vdash 3$ and the value of (6.21) is

$$\begin{aligned}n^2 - (n-3)^2 - \lambda_{d-2}^2 + 2(d-3)\lambda_{d-2} - \lambda_{d-1}^2 + 2(d-2)\lambda_{d-1} - \lambda_d^2 + 2(d-1)\lambda_d \\ = 6(n+d-4).\end{aligned} \tag{6.16}$$

If $d < 1 + \lfloor\sqrt{k}\rfloor$, then we show that the maximum is attained at any partition λ of height d with $\lambda_1 \geq k$ whose tail is a balanced partition of k of largest possible height $d-e$ (which has to be $< \lfloor\sqrt{k}\rfloor$). It will follow that to maximize (6.13), we must set $e = 1$. Then

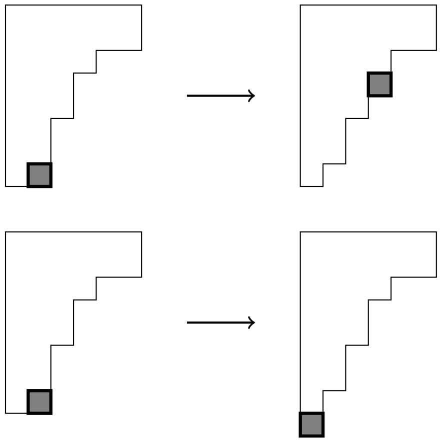


FIGURE 1. Dagger operations
When tracking down a pair of partitions λ, μ that maximizes $\eta_\lambda - \eta_\mu$ in the proof of Theorem 6.16, we repeatedly rearrange λ or μ into $\lambda^\dagger$ or $\mu^\dagger$ by moving the last box to either an earlier row (a) or a new row (b), and show that this operation strictly increases $\eta_\lambda - \eta_\mu$.

$(\lambda_2, \dots, \lambda_d)$ is a balanced partition of k. Formula (6.11) in part (2) is again obtained by plugging the maximizing λ into (6.13) using the expression (6.4). ∎

Lemma 6.17. *The expression* (6.13) *is maximized at partitions $\lambda \vdash n$ and $\mu \vdash n-k$ such that the rows of μ coincide with the rows of λ. This means that $\mu_i = \lambda_i$ for $i = 1, \dots, \mathrm{ht}(\mu)$.*

Proof. Let $\lambda \vdash n$ and $\mu \vdash n-k$ be such that λ is obtained from μ by adding k boxes. First consider the case when $\lambda_1 > n-k$. Let $\mu^\dagger = \mu$ and construct $\lambda^\dagger$ from λ by moving the last box of λ_1 (which is not part of μ as $\lambda_1 > n-k$ and $k \le n/2$) to the second row of λ. Note that $\lambda_2 < \lambda_1$ since $\lambda_1 > n-k$ and $k \le n/2$, so $\lambda^\dagger$ is in fact a valid partition. Then $\lambda_1^\dagger = \lambda_1 - 1, \lambda_2^\dagger = \lambda_2 + 1$ and

$$\begin{aligned}\eta_\lambda - \eta_\mu - (\eta_\lambda - \eta_{\mu^\dagger}) &= -\lambda_1^2 + (\lambda_1^\dagger)^2 - \lambda_2^2 + 2\lambda_2 + (\lambda_2^\dagger)^2 - 2\lambda_2^\dagger \\ &= 2(\lambda_2 - \lambda_1) \le 0.\end{aligned}$$

Hence, it remains to consider partitions λ with $\lambda_1 \le n-k$. In this case let j_0 be the smallest index such that $\lambda_{j_0} > \mu_{j_0}$ (hence $\lambda_j = \mu_j$ for all $j = 1, \dots, j_0 - 1$). Note that $\lambda_1 \le n-k$ implies that $j_0 \ge 2$. Now construct a partition $\mu^\dagger \vdash n-k$ from μ by moving the last box of the last (e-th) row of μ to the j_0-th row of μ. Hence, $\mu_{j_0}^\dagger = \mu_{j_0} + 1$ and $\mu_e^\dagger = \mu_e - 1$. Note that by the definition of j_0, this construction really gives a (valid) partition, i.e., $\mu_{j_0}^\dagger \le \mu_{j_0-1}^\dagger$ and $\mu_{j_0}^\dagger \le \lambda_{j_0}$. So if $\mu_e > 1$,

$$\begin{aligned}\eta_\lambda - \eta_\mu - (\eta_\lambda - \eta_{\mu^\dagger}) &= \mu_{j_0}^2 - (\mu_{j_0}^\dagger)^2 + (\mu_e - e + 1)^2 - (\mu_e^\dagger - e + 1)^2 \\ &= -2(e + \mu_{j_0} - \mu_e) < 0.\end{aligned}$$

Otherwise, if $\mu_e = 1$, then $\mu^\dagger$ has $e-1$ rows and we get

$$\begin{aligned}\eta_\lambda - \eta_\mu - (\eta_\lambda - \eta_{\mu^\dagger}) &= \mu_{j_0}^2 - (\mu_{j_0}^\dagger)^2 + (\mu_e - e + 1)^2 \\ &\quad - \frac{e(e-1)(2e-1)}{6} + \frac{(e-1)(e-2)(2e-3)}{6} \\ &= -2(e-1) - 2\mu_{j_0} < 0.\end{aligned}$$

We deduce that the maximal value of (6.13) is obtained among the pairs of partitions (λ, μ) with $\lambda_j = \mu_j$ for $j = 1, \dots, e$ so that the rows $\lambda_{e+1}, \lambda_{e+2}, \dots, \lambda_f$ form a partition of k. ∎

Example 6.18. In Lemma 6.17, we do not claim that all pairs of partitions (λ, μ) that maximize (6.13) satisfy $\mu_i = \lambda_i$ for $i = 1, \dots, \mathrm{ht}(\mu)$. In fact, when $n = 11$ and $k = d = 5$, the maximum is attained at partitions $(6, 2, 1, 1, 1)$ and $(5, 2, 2, 1, 1)$, and no tail of the latter is a partition of 5.

Lemma 6.19. *Assume $1 + \lfloor\sqrt{k}\rfloor \le d \le n - k + 1$ and consider pairs of partitions $\lambda \vdash n, \mu \vdash n - k$ with $\lambda_1 \ge k$ such that the rows of μ coincide with the first $e = \mathrm{ht}(\mu)$ rows of λ. Among such pairs of partitions, the maximal value of (6.13) is attained when the height of λ is d.*

Proof. If $\lambda \vdash n$ has height $f < d$, let l_0 be the biggest index such that $\lambda_{l_0} > 1$. The partition $\lambda^\dagger \vdash n$ is obtained from λ by moving the last box of the l_0-th row of λ to a new row (so that $\lambda^\dagger$ has $f + 1$ rows). Note that the assumption $d \le n - k + 1$ guarantees that assuming $l_0 = 1$ and $\lambda_1 = k$ leads to the contradiction $f = d$. Thus $l_0 = 1$ implies $\lambda_1 > k$ so that $\lambda_1^\dagger \ge k$ as well.

Let $\mu = (\lambda_1, \dots, \lambda_e) \vdash n - k$. If $l_0 > e$, then set $\mu^\dagger = \mu$ and compute

$$\begin{aligned}\eta_\lambda - \eta_\mu - (\eta_{\lambda^\dagger} - \eta_\mu) = \eta_\lambda - \eta_{\lambda^\dagger} &= \frac{f(f-1)(2f-1)}{6} - \frac{(f+1)f(2f+1)}{6} \\ &\quad - (\lambda_{l_0} - l_0 + 1)^2 + (\lambda^\dagger_{l_0} - l_0 + 1)^2 + (\lambda^\dagger_{f+1} - f)^2 \\ &= -2(\lambda_{l_0} + f - l_0) < 0.\end{aligned}$$

If $l_0 \le e$, then $\mu_{l_0} = \lambda_{l_0}$ and $\mu^\dagger$ is constructed from μ by moving the last box of the l_0-th row of μ to a new row (so that μ has $e + 1$ rows). Note that by the definition of l_0, we indeed obtain a valid partition. Hence,

$$\begin{aligned}\eta_\lambda - \eta_\mu - (\eta_{\lambda^\dagger} - \eta_{\mu^\dagger}) &= \frac{f(f-1)(2f-1)}{6} - \frac{(f+1)f(2f+1)}{6} \\ &\quad - \frac{e(e-1)(2e-1)}{6} + \frac{(e+1)e(2e+1)}{6} \\ &\quad - (\lambda_{l_0} - l_0 + 1)^2 + (\lambda^\dagger_{l_0} - l_0 + 1)^2 + (\lambda^\dagger_{f+1} - f)^2 \\ &\quad + (\mu_{l_0} - l_0 + 1)^2 - (\mu^\dagger_{l_0} - l_0 + 1)^2 - (\mu^\dagger_{e+1} - e)^2 \\ &= -2(f - e) < 0.\end{aligned}$$

■

Lemma 6.20. *Assume $1 + \lfloor\sqrt{k}\rfloor \le d \le n - k + 1$ and consider pairs of partitions $\lambda \vdash n, \mu \vdash n - k$ with $\lambda_1 \ge k$ such that the rows of μ coincide with the first $e = \mathrm{ht}(\mu)$ rows of λ. Among such pairs of partitions, the maximal value of (6.13) is attained when the tail $(\lambda_{e+1}, \dots, \lambda_d)$ of λ is a balanced partition of k.*

Proof. By Lemma 6.19, the height of λ is d. Then

$$\begin{aligned}\eta_\lambda - \eta_\mu &= n^2 - (n-k)^2 + \frac{d(d-1)(2d-1)}{6} \\ &\quad - \frac{e(e-1)(2e-1)}{6} - \sum_{i=e+1}^{d} (\lambda_i - i + 1)^2.\end{aligned} \tag{6.17}$$

To maximize (6.17), we have to minimize

$$\sum_{i=e+1}^{d} (\lambda_i - i + 1)^2.$$

This expression attains its minimum at any partition $(\lambda_{e+1},\dots,\lambda_d)\vdash k$ that is balanced (cf. the proof of Corollary 6.8), i.e., satisfying $\lambda_d-\lambda_{e+1}\le 1$ and is hence of the form

$$(\lambda_{e+1},\dots,\lambda_d)=\Big(\underbrace{1+\frac{k-r}{d-e},\dots,1+\frac{k-r}{d-e}}_{r},\ \underbrace{\frac{k-r}{d-e},\dots,\frac{k-r}{d-e}}_{d-e-r}\Big) \tag{6.18}$$

as in (6.5), where n is replaced by k, and d is replaced by $d-e$, thus $r\equiv k \pmod{d-e}$. ■

Lemma 6.21. *Let $\lambda^1,\lambda^2\vdash n$ be two partitions of the same height f with $\lambda^1_1,\lambda^2_1\ge k$ and with the same balanced tails of height $f-e$. Let $\mu^1=(\lambda^1_1\dots,\lambda^1_e)\vdash n-k$ and $\mu^2=(\lambda^2_1,\dots,\lambda^2_e)\vdash n-k$. Then $\eta_{\lambda^1}-\eta_{\mu^1}=\eta_{\lambda^2}-\eta_{\mu^2}$.*

Proof. Let $\lambda\vdash n$ be any partition whose tail $\xi=(\lambda_{e+1},\dots,\lambda_f)$ is a balanced partition of k. To prove the claim, it suffices to show that removing any box from $\mu=(\lambda_1,\dots,\lambda_e)$ and placing it at the end of any other row of μ does not change the value of (6.13).

If $e=1$, i.e., $\lambda_1=n-k$, then there is the only one partition $\lambda\vdash n$ with the first row $\ge k$ and the given balanced tail. Hence, assume $e\ge 2$ and let $1\le a,b\le e$ be distinct. Assume that moving one box from the a-th row to the b-th row of λ (and hence of μ) produces a valid partition $\lambda^\dagger$ (and $\mu^\dagger$). We show that in this case

$$\eta_\lambda-\eta_\mu=\eta_{\lambda^\dagger}-\eta_{\mu^\dagger}.$$

Indeed,

$$\begin{aligned}\eta_\lambda-\eta_\mu-\eta_{\lambda^\dagger}+\eta_{\mu^\dagger}&=-(\lambda_a-a+1)^2+(\mu_a-a+1)^2+(\lambda^\dagger_a-a+1)^2-(\mu^\dagger_a-a+1)^2\\&\quad-(\lambda_b-b+1)^2+(\mu_b-b+1)^2+(\lambda^\dagger_b-b+1)^2-(\mu^\dagger_b-b+1)^2\\&=0\end{aligned}$$

since $\lambda_a=\mu_a$ and $\lambda^\dagger_a=\mu^\dagger_a$. ■

Lemma 6.22. *Assume $1+\lfloor\sqrt{k}\rfloor\le d\le n-k+1$. Let $\lambda\vdash n$ be a partition of height d with $\lambda_1\ge k$ and a balanced tail of smallest possible height g and let $\mu=(\lambda_1,\dots,\lambda_{d-g})$. If $g\ge\lfloor\sqrt{k}\rfloor$, then any partition $\lambda^\dagger\vdash n$ of height d with a balanced tail of height $g^\dagger>g$ together with $\mu^\dagger=(\lambda_1,\dots,\lambda_{d-g^\dagger})$ gives rise to at most a lower value of (6.13).*

Proof. Let $\lambda\vdash n$ be any partition with $\lambda_1\ge k$ whose tail $\xi=(\lambda_{e+1},\dots,\lambda_d)$ is a balanced partition of k of the smallest possible height $d-e$ and assume $d-e\ge\lfloor\sqrt{k}\rfloor$. We can assume $\lambda_{e+1}>1$, since otherwise there is no partition with a taller tail. Hence the following construction gives a valid partition of n. Let $\lambda^\dagger$ be the partition obtained from λ by replacing ξ (which has $d-e$ rows) by a balanced partition on $d-e+1$ rows and by removing the e-th row of λ and adding it to the first row. To obtain $\mu^\dagger$, remove the e-th row from μ and add it to the first row (so that $\mathrm{ht}(\mu^\dagger)=\mathrm{ht}(\mu)-1$). Since $d-e\ge\lfloor\sqrt{k}\rfloor$, we have $\lambda^\dagger_i=\lambda_{i+1}-\epsilon_i$ with $\epsilon_i\in\{0,1\}$ for $i=e,\dots,d-1$. Note that there are precisely $\lambda^\dagger_d$ many indices i such that $\epsilon_i=1$. Then

$$\begin{aligned}\eta_\lambda-\eta_\mu-\eta_{\lambda^\dagger}+\eta_{\mu^\dagger}&=-\lambda_1^2+(\lambda^\dagger_1)^2-\sum_{i=e}^{d}\lambda_i^2-(\lambda^\dagger_i)^2-2(i-1)(\lambda_i-\lambda^\dagger_i)\\&\quad+\mu_1^2-(\mu^\dagger_1)^2+\mu_e^2-2(e-1)\mu_e\\&=(\lambda^\dagger_d)^2-2(d-1)\lambda^\dagger_d-\sum_{i=e}^{d-1}\lambda_{i+1}^2-(\lambda^\dagger_i)^2-2i\lambda_{i+1}+2(i-1)\lambda^\dagger_i\end{aligned}$$

$$= (\lambda_d^\dagger)^2 - 2(d-1)\lambda_d^\dagger + \sum_{i=e}^{d-1} \epsilon_i^2 + 2\epsilon_i(-1+i-\lambda_{i+1}) + 2\lambda_{i+1}$$
$$= (\lambda_d^\dagger)^2 - 2(d-1)\lambda_d^\dagger + \sum_{\epsilon_i=1}(2i-1) + \sum_{\epsilon_i=0} 2\lambda_{i+1}.$$

It is easy to see that the sum over all indices i with $\epsilon_i = 1$ has $\lambda_d^\dagger$ terms (and the complemental sum has $d-e-\lambda_d^\dagger$ terms), hence

$$\begin{aligned}
&(\lambda_d^\dagger)^2 - 2(d-1)\lambda_d^\dagger + \sum_{\epsilon_i=1}(2i-1) + \sum_{\epsilon_i=0} 2\lambda_{i+1} \\
&= (\lambda_d^\dagger)^2 + \lambda_d^\dagger + \sum_{\epsilon_i=1} 2(i-d) + \sum_{\epsilon_i=0} 2\lambda_{i+1} \\
&\geq (\lambda_d^\dagger)^2 + \lambda_d^\dagger + \sum_{\epsilon_i=0} 2\lambda_{i+1} - 2\left[d-e+1-\lambda_d^\dagger + \cdots + d-e\right] \\
&= (\lambda_d^\dagger)^2 + \lambda_d^\dagger - (1+2(d-e)-\lambda_d^\dagger)\lambda_d^\dagger + \sum_{\epsilon_i=0} 2\lambda_{i+1} \\
&= 2(\lambda_d^\dagger)^2 - 2(d-e)\lambda_d^\dagger + \sum_{\epsilon_i=0} 2\lambda_{i+1} \\
&= 2(\lambda_d^\dagger)^2 - 2(d-e-\lambda_d^\dagger)\lambda_d^\dagger - 2(\lambda_d^\dagger)^2 + \sum_{\epsilon_i=0} 2\lambda_{i+1} \\
&= \sum_{\epsilon_i=0} 2(\lambda_{i+1} - \lambda_d^\dagger) \geq 0.
\end{aligned}$$

By Lemma 6.21, all partitions (with first part $\geq k$ and) with tail of length $d-e+1$ give the same value (6.13), from which the claim follows. ∎

Lemma 6.23. *Assume $d \leq n-k+1$. Let $\lambda \vdash n$ be a partition of height d with $\lambda_1 \geq k$ and a balanced tail of smallest possible height g and let $\mu = (\lambda_1, \ldots, \lambda_{d-g})$. If $g < \lfloor\sqrt{k}\rfloor$, then any partition $\lambda^\dagger \vdash n$ of height d with a balanced tail of height $g^\dagger > g$ together with $\mu^\dagger = (\lambda_1, \ldots, \lambda_{d-g^\dagger})$ gives rise to a larger value of* (6.13).

Proof. Let $\lambda \vdash n$ be any partition of height d with $\lambda_1 \geq k$ whose tail $\xi = (\lambda_{e+1}, \ldots, \lambda_d)$ is a balanced partition of k of the smallest possible height $d-e$. Assume $d-e < \lfloor\sqrt{k}\rfloor$. Now construct $\lambda^\dagger, \mu^\dagger$ from λ and $\mu = (\lambda_1, \ldots, \lambda_e)$ as in the proof of Lemma 6.22. This time we have $\lambda_i^\dagger = \lambda_{i+1} - \epsilon_i$, where $\epsilon_i > 1$ for at least one index i. However, the sum of the ϵ_i is still $\lambda_d^\dagger$. Hence

$$\begin{aligned}
\eta_\lambda - \eta_\mu - \eta_{\lambda^\dagger} + \eta_{\mu^\dagger} &= -\lambda_1^2 + (\lambda_1^\dagger)^2 - \sum_{i=e}^{d} \lambda_i^2 - (\lambda_i^\dagger)^2 - 2(i-1)(\lambda_i - \lambda_i^\dagger) \\
&\quad + \mu_1^2 - (\mu_1^\dagger)^2 + \mu_e^2 - 2(e-1)\mu_e \\
&= (\lambda_d^\dagger)^2 - 2(d-1)\lambda_d^\dagger - \sum_{i=e}^{d-1} \lambda_{i+1}^2 - (\lambda_i^\dagger)^2 - 2i\lambda_{i+1} + 2(i-1)\lambda_i^\dagger \\
&= (\lambda_d^\dagger)^2 - 2(d-1)\lambda_d^\dagger + \sum_{i=e}^{d-1} \epsilon_i^2 + 2\epsilon_i(i-1) + 2\lambda_{i+1}(1-\epsilon_i)
\end{aligned}$$

$$=(\lambda_d^\dagger)^2+\sum_{i=e}^{d-1}\epsilon_i^2+2\epsilon_i(i-d)+2\lambda_{i+1}(1-\epsilon_i)$$
$$=\sum_{i=e}^{d-1}\epsilon_i^2+\epsilon_i\lambda_d^\dagger+2\epsilon_i(i-d)+2\lambda_{i+1}(1-\epsilon_i)$$

Since $\lambda_{i+1}\geq\lambda_d^\dagger+\epsilon_i$, we get

$$\begin{aligned}\sum_{i=e}^{d-1}\epsilon_i^2+\epsilon_i\lambda_d^\dagger+2\epsilon_i(i-d)+2\lambda_{i+1}(1-\epsilon_i) &\leq \sum_{i=e}^{d-1}\epsilon_i^2+\epsilon_i\lambda_d^\dagger+2\epsilon_i(i-d)+2(\lambda_d^\dagger+\epsilon_i)(1-\epsilon_i)\\ &=\sum_{\epsilon_i\geq 2}-\epsilon_i^2+2\epsilon_i+\epsilon_i\lambda_d^\dagger+2\epsilon_i(i-d)+2\lambda_d^\dagger(1-\epsilon_i)\\ &\quad+\sum_{\epsilon_i=1}1+\lambda_d^\dagger+2(i-d)\end{aligned}$$

Since the tails of λ and $\lambda^\dagger$ are balanced, the ϵ_i can only take two distinct values. If $\epsilon_i\geq 2$ for all i, then $\eta_\lambda-\eta_\mu-\eta_{\lambda^\dagger}+\eta_{\mu^\dagger}$ is clearly negative by the above. On the other hand, if $\epsilon_i\in\{1,2\}$, note that $\lambda_d^\dagger=d-e+\alpha$, where $\alpha=|\{\epsilon_i=2\}|$ and we have

$$\begin{aligned}\eta_\lambda-\eta_\mu-\eta_{\lambda^\dagger}+\eta_{\mu^\dagger} &\leq \sum_{i=e}^{d-1}-\epsilon_i^2+2\epsilon_i+\epsilon_i\lambda_d^\dagger+2\epsilon_i(i-d)+2\lambda_d^\dagger(1-\epsilon_i)\\ &=-4\sum_{\epsilon_i=2}d-i+\sum_{\epsilon_i=1}1+\lambda_d^\dagger-2(d-i)\\ &=-4\sum_{\epsilon_i=2}d-i+\sum_{\epsilon_i=1}1+d-e+\alpha-2(d-i)\\ &\leq -4\big((d-e-\alpha+1)+\cdots+(d-e)\big)+\\ &\quad+(1+d-e+\alpha)(\lambda_d^\dagger-\alpha)-2(1+\cdots+\lambda_d^\dagger-\alpha)\\ &=-2(d-e)(d-e+1)+2(d-e-\alpha)(d-e-\alpha+1)\\ &\quad+(1+d-e+\alpha)(\lambda_d^\dagger-\alpha)-(\lambda_d^\dagger-\alpha)(\lambda_d^\dagger-\alpha+1)\\ &=-2(d-e)(d-e+1)+2(d-e-\alpha)(d-e-\alpha+1)\\ &\quad+(d-e+2\alpha-\lambda_d^\dagger))(\lambda_d^\dagger-\alpha)\\ &=\alpha(-2+2\alpha-3(d-e))\leq 0,\end{aligned}$$

since $\alpha\leq d-e$. ■

Corollary 6.24.

(1) *Let $1+\lfloor\sqrt{k}\rfloor\leq d\leq n-k+1$. Among partitions $\lambda\vdash n$ with $\lambda_1\geq k$, the maximal value of (6.13) is attained at any partition λ of height d whose tail is a balanced partition of k of smallest possible height $\geq\lfloor\sqrt{k}\rfloor$.*

(2) *Let $d<1+\lfloor\sqrt{k}\rfloor$. Among partitions $\lambda\vdash n$ with $\lambda_1\geq k$, the maximal value of (6.13) is attained at the partition $\lambda\vdash n$ with d rows and $\lambda_1=n-k$ such that $(\lambda_2,\dots,\lambda_d)$ is a balanced partition of k.*

Proof. The proof goes along the lines of the proof of Theorem 6.16. ■

Remark 6.25. Let $\xi = (\lambda_{e+1}, \dots, \lambda_d) \vdash k$ be the (balanced) tail of $\lambda \vdash n$ and let $\xi^\dagger \vdash k$ be the (balanced) tail of $\lambda^\dagger$, i.e., $\xi^\dagger$ is the balanced partition of k with $d-e+1$ rows. Let $\lambda_i^\dagger = \lambda_{i+1} - \epsilon_i$ for $i = e, \dots, d-1$. Note that the following holds:

(1) If $d - e > \lfloor\sqrt{k}\rfloor$, then $d - e \geq \lambda_{e+1}$, i.e., ξ looks roughly like a tall rectangle. In this case clearly $\epsilon_i \leq 1$ for all i and by Lemma 6.22, lowering e produces at most a lower value of (6.13) (assuming $\lambda_1 \geq k$).

(2) If $d - e = \lfloor\sqrt{k}\rfloor$, then
$$\lambda_{e+1} \leq \frac{(\lfloor\sqrt{k}\rfloor + 1)^2 - 1}{\lfloor\sqrt{k}\rfloor} = \lfloor\sqrt{k}\rfloor + 2.$$
In this case the last column of ξ becomes $\lambda_d^\dagger$ and $\lambda_e^\dagger \leq \lfloor\sqrt{k}\rfloor + 1$. Hence, $\epsilon_i \leq 1$ for all i again, and by Lemma 6.22, lowering e produces at most a lower value of (6.13) (assuming $\lambda_1 \geq k$).

(3) If $d - e < \lfloor\sqrt{k}\rfloor$, then ξ roughly looks like a wide rectangle. Then there is an i such that
$$\lambda_i \geq d - e + 2.$$
Hence, there is an i such that $\epsilon_i \geq 2$. Then by Lemma 6.23, lowering e produces a larger value of (6.13).

Example 6.26. Consider the d-QMC Hamiltonian for $K_{n-2,2}$ with $n \geq 4$. In this case the condition $(k-1)d < n$ becomes $d < n$, which is what we always assume in avoidance of trivial cases. Further assuming the nontrivial case $d \geq 2 = 1 + \lfloor\sqrt{2}\rfloor$, all the conditions of Theorem 6.16 are met. Hence, the solution of the d-QMC problem for $K_{n-2,2}$ is

$$4(n + d - 3). \tag{6.19}$$

If $d \geq n$, then the d-QMC problem is trivial, cf. Subsection 6.1. In this case, the solution to the d-QMC problem is $8(n-2)$ (since $K_{n-2,2}$ has $2(n-2)$ edges), so (6.19) is only valid for $d < n$.

Example 6.27. Consider $K_{n-3,3}$ for $n \geq 6$. The condition $(k-1)d < n$ becomes $2d < n$ and the second condition $d \geq 2 = \lfloor\sqrt{3}\rfloor + 1$ can be assumed w.l.o.g. as for the case $k = 2$ of Example 6.26. Hence, by Theorem 6.16(1), the solution to the d-QMC problem for $K_{n-3,3}$ with $2d < n$ equals

$$6(n + d - 4), \tag{6.20}$$

attained at any $\lambda \vdash n$ of height d with $\lambda_1 \geq 3$ and tail of smallest possible height. We shall prove that (6.20) holds unconditionally, i.e., for all n, k, d with $d < n$ and $2k \leq n$.

To solve the d-QMC problem for $K_{n-3,3}$, we are maximizing

$$\eta_\lambda - \eta_\mu - \eta_\nu, \tag{6.21}$$

where $\lambda \vdash n$, $\mu \vdash n-3$ is obtained from λ by removing 3 boxes and $\nu \vdash 3$ is obtained from λ by removing $n-3$ boxes. Let $e = \mathrm{ht}(\mu)$. We separate two cases:

(a) Consider partitions $\lambda \vdash n$ with $\lambda_1 \geq 3$. Then $d \leq n-2$, and Corollary 6.24 applies; the maximal value of (6.21) is attained when $\nu = (3)$ and $\lambda \vdash n$ has tail of smallest possible height. Now (6.14), (6.15) and (6.16) show that in this case, the maximal value equals (6.20) and is thus independent of the height of the tail.

(b) Now consider partitions $\lambda \vdash n$ with $\lambda_1 = 2$. Since $\eta_{(2,1)} = 6$ and $\eta_{(1,1,1)} = 12$, the maximal value of (6.21) is attained with $\nu = (2,1) \vdash 3$. We are thus essentially again maximizing (6.13), whence Lemma 6.17 applies. Note that the tail of λ can only have height 2 or 3.

Suppose $\operatorname{ht}(\lambda) = f \le d$. If the tail is $(2,1) \vdash 3$, then (6.21) becomes

$$n^2 - (n-3)^2 - \lambda_{f-1}^2 + 2(f-2)\lambda_{f-1} - \lambda_f^2 + 2(f-1)\lambda_f - 6 \\ = 6(n+f-5) \le 6(n+d-5).$$

If the tail is $(1,1,1) \vdash 3$, then (6.21) becomes

$$n^2 - (n-3)^2 - \lambda_{f-2}^2 + 2(f-3)\lambda_{f-2} - \lambda_{f-1}^2 + 2(f-2)\lambda_{f-1} - \lambda_f^2 + 2(f-1)\lambda_f - 6 \\ = 6(n+f-5) \le 6(n+d-5).$$

Note that we can omit the case $\lambda_1 = 1$, since it implies that $d = n$. It is now clear that the maximal value of (6.21) is $6(n+d-4)$.

Example 6.28. Although (6.10) unconditionally solves the d-QMC problem for both $K_{n-2,2}$ and $K_{n-3,3}$, the formula finally breaks for $n = 8$, $k = d = 4$. Here, (6.10) yields the value 60, while a brute force search over all partitions λ of 8 with height ≤ 4 shows that solution to the 4-QMC problem for $K_{4,4}$ is 56 attained at $\lambda = (4,2,1,1) \vdash 8$. Equation (6.10) also fails for $d > 4$, where the solution to the d-QMC problem for $K_{4,4}$ is 64, attained at one of the following partitions: $(4,1,1,1,1)$, $(3,1,1,1,1,1)$, $(2,1,1,1,1,1,1)$, $(1,1,1,1,1,1,1,1)$.

We conjecture that Corollary 6.24 holds unconditionally, that is, even without assuming $\lambda_1 \ge k$. Moreover, we conjecture that if $d > n-k+1$, then the solution to the d-QMC for $K_{n-k,k}$ is attained at the partition $\lambda \vdash n$ with $n-k+1$ rows such that $\lambda_1 = k$ and $\lambda_j = 1$ for $j \ge 1$.

7. Separation of irreps in d-QMC

Given $n, d \in \mathbb{N}$ and a partition $\lambda \vdash n$ with at most d rows, we are interested in adapting the NPO hierarchy in Section 5 to compute the largest eigenvalue of H_G^λ for a general weighted graph G on n vertices. To isolate the irreducible representation ρ_λ of $\mathcal{A}_n^{\mathrm{Sw}_d} = \mathcal{F}_n/\mathcal{I}_n^{\mathrm{Sw}_d}$ corresponding to the partition λ of n, one needs to adjoin to $\mathcal{I}_n^{\mathrm{Sw}_d}$ some polynomials in $\mathcal{F}_n$ that vanish in $\rho_\lambda(\mathbb{C}S_n)$, but not in $\rho_\mu(\mathbb{C}S_n)$ for any $\mu \neq \lambda$.

In general it suffices a add a single polynomial, chosen as follows. Given a partition $\lambda \vdash n$ let $s_\lambda \in \mathbb{C}S_n$ be a Young symmetrizer corresponding to λ [Pro07, Section 9.2.2]. Then $\frac{\dim \rho_\lambda}{n!} s_\lambda$ is a primitive idempotent in $\mathbb{C}S_n$ that generates ρ_n as a left ideal [Pro07, Theorems 9.2.4.1 and 9.2.4.2]. Hence

$$\hat{s}_\lambda = \frac{\dim \rho_\lambda}{(n!)^2} \sum_{\sigma \in S_n} \sigma s_\lambda \sigma^{-1}$$

is a centrally primitive idempotent in $\mathbb{C}S_n$, generating ρ_λ as a two-sided ideal.

Proposition 7.1. *Let $\lambda \vdash n$. Then $\rho_\lambda(\mathrm{id} - \hat{s}_\lambda) = 0$ and $\rho_\mu(\mathrm{id} - \hat{s}_\lambda) \neq 0$ every partition $\mu \neq \lambda$.*

Proof. The centrally primitive idempotents $\{\hat{s}_\lambda\}_{\lambda \vdash n}$ are pairwise orthogonal, so $(\mathrm{id} - \hat{s}_\lambda)\hat{s}_\mu = 0$ if $\lambda = \mu$ and $\hat{s}_\mu$ otherwise. Therefore $\rho_\lambda(\mathrm{id} - \hat{s}_\lambda) = 0$ and $\rho_\mu(\mathrm{id} - \hat{s}_\lambda) \neq 0$ for $\mu \neq \lambda$. ■

Lifting $\hat{s}_\lambda$ to an element of $\mathcal{F}_n$ yields the desired polynomial. However, this polynomial has high degree (not much smaller than n). On the other hand, the symmetric group relations (1.4) have degree at most 3, and the degree-reducing antisymmetrizer relation (3.1) has degree d. Therefore the above approach is not appealing from a computational perspective. Instead, it is preferable to find low-degree polynomials that distinguish λ from other partitions of n with at most d rows.

In [BCEHK24] it was shown that for $d = 2$, the value η_λ from Example 6.6 separates irreps with at most two rows. Therefore, $\eta_\lambda - h_{K_n}$ is a linear polynomial that separates irreps of $\mathcal{A}_n^{\mathrm{Sw}_2}$. In particular, the largest eigenvalue of H_G^λ for a two-row partition $\lambda \vdash n$ equals the NPO problem

$$\min \left\{ \alpha : \alpha - h_G = \sum_k s_k^* s_k + q \text{ for some } s_k \in \mathcal{F}_n,\ q \in \mathcal{I}_n^{\mathrm{Sw}_d} + \left(\eta_\lambda - h_{K_n}\right) \right\},$$

and can thus be handled using standard SDP-based NPO hierarchies.

The same does not apply when $d = 3$, as η_λ in (6.4) does not separate irreps with at most three rows. For example, partitions $\lambda = (4, 1, 1)$ and $\mu = (3, 3)$ of $n = 6$ give $\eta_\lambda = \eta_\mu = 24$. Even more, η_λ does not separate irreps with three rows; e.g., $\lambda = (5, 2, 2)$ and $\mu = (4, 4, 1)$ give $\eta_\lambda = \eta_\mu = 60$. Below, we present a method of separating irreps with at most three rows in the spirit of 3-QMC, and a method of separating general irreps that is suitable for solving the localized d-QMC problem of finding the largest eigenvalue of H_G^λ.

7.1. **Separation of irreps with at most three rows via two graphs.** First we show that the spectra of the Hamiltonians corresponding to the clique K_n and the star graph $\bigstar_n$ (which were analyzed in Subsections 6.2 and 6.3.1, respectively) distinguish partitions with at most three rows.

Proposition 7.2. *Let $n \geq 2$. The following are equivalent for partitions $\lambda, \mu \vdash n$ with at most three rows:*

(i) $\mathrm{spec}(H_{K_n}^\lambda) = \mathrm{spec}(H_{K_n}^\mu)$ *and* $\mathrm{spec}(H_{\bigstar_n}^\lambda) \subseteq \mathrm{spec}(H_{\bigstar_n}^\mu)$*;*
(ii) $\mathrm{spec}(H_{\bigstar_n}^\lambda) = \mathrm{spec}(H_{\bigstar_n}^\mu)$*;*
(iii) $\lambda = \mu$.

Proof. It is clear that (iii) implies both (i) and (ii).

(i)⇒(iii): By Example 6.10 and Example 6.12, the eigenvalues of $nI - \frac{1}{2}H_{\bigstar_n}^\lambda$ are obtained from the sequence $\lambda_1 > \lambda_2 - 1 > \lambda_3 - 2$ by keeping only the smallest element of any subsequence of consecutive values, and then removing -2 if necessary. Consequently,

$$\lambda_1 + (\lambda_2 - 1) + (\lambda_3 - 2) = n - 3 = \mu_1 + (\mu_2 - 1) + (\mu_3 - 2) \tag{7.1}$$

and $\mathrm{spec}(H_{\bigstar_n}^\lambda) \subseteq \mathrm{spec}(H_{\bigstar_n}^\mu)$ immediately imply $\lambda = \mu$ if $|\mathrm{spec}(H_{\bigstar_n}^\lambda)| \geq 2$. Now assume $|\mathrm{spec}(H_{\bigstar_n}^\lambda)| = 1$. By Proposition 6.7, $\mathrm{spec}(H_{K_n}^\lambda) = \mathrm{spec}(H_{K_n}^\mu)$ implies

$$\lambda_1^2 + (\lambda_2 - 1)^2 + (\lambda_3 - 2)^2 = \mu_1^2 + (\mu_2 - 1)^2 + (\mu_3 - 2)^2 \tag{7.2}$$

We distinguish three cases:

(1) $\lambda = (\frac{n}{3}, \frac{n}{3}, \frac{n}{3})$, and the sole eigenvalue of $nI - \frac{1}{2}H_{\bigstar_n}^\lambda$ is $\frac{n}{3} - 2$. Since $\mu_1 \geq \frac{n}{3}$, if follows that $\mu_2 - 1 = \frac{n}{3} - 2$ or $\mu_3 - 2 = \frac{n}{3} - 2$. In the latter case $\mu = \lambda$, so let us suppose the former holds. Then (7.1) and (7.2) imply

$$n - 3 = \mu_1 + (\tfrac{n}{3} - 2) + (\mu_3 - 2),$$
$$(\tfrac{n}{3})^2 + (\tfrac{n}{3} - 1)^2 + (\tfrac{n}{3} - 2)^2 = \mu_1^2 + (\tfrac{n}{3} - 2)^2 + (\mu_3 - 2)^2.$$

Expressing $\mu_3 = \frac{2n}{3} + 1 - \mu_1$ gives

$$(\tfrac{n}{3})^2 + (\tfrac{n}{3} - 1)^2 = \mu_1^2 + (\tfrac{2n}{3} - \mu_1 - 1)^2,$$

which has solutions $\mu_1 = \frac{n}{3}$ and $\mu_1 = \frac{n}{3} - 1$. The first one implies $\mu = \lambda$, while the second one contradicts the fact that μ is a partition of n.

(2) $\lambda = (\frac{n}{2}, \frac{n}{2})$, and the sole eigenvalue of $nI - \frac{1}{2}H^{\lambda}_{\bigstar_n}$ is $\frac{n}{2} - 1$. Since $\mu_3 < \frac{n}{2}$, it follows that $\mu_1 = \frac{n}{2} - 1$ or $\mu_2 - 1 = \frac{n}{2} - 1$. In the latter case $\mu = \lambda$, so let us assume the former holds. Then (7.1) and (7.2) imply

$$n - 3 = (\tfrac{n}{2} - 1) + (\mu_2 - 1) + (\mu_3 - 2),$$
$$(\tfrac{n}{2})^2 + (\tfrac{n}{2} - 1)^2 + (0 - 2)^2 = (\tfrac{n}{2} - 1)^2 + (\mu_2 - 1)^2 + (\mu_3 - 2)^2.$$

Expressing $\mu_2 = \frac{n}{2} + 1 - \mu_3$ gives

$$(\tfrac{n}{2})^2 + 4 = (\tfrac{n}{2} - \mu_3)^2 + (\mu_3 - 2)^2,$$

which has solutions $\mu_3 = 0$ and $\mu_3 = \frac{n}{2} + 2$. The first one implies $\mu = \lambda$, while the second one contradicts the fact that μ is a partition of n.

(3) $\lambda = (n)$, and the sole eigenvalue of $nI - \frac{1}{2}H^{\lambda}_{\bigstar_n}$ is n. Then $\mu_3 \le \mu_2 \le n$ implies $\mu_1 = n$, and so $\mu = \lambda$.

(ii)⇒(iii): As in the previous paragraph we see that (ii) and (7.1) imply $\lambda = \mu$ if $|\operatorname{spec}(H^{\lambda}_{\bigstar_n})| \ge 2$. On the other hand, if $|\operatorname{spec}(H^{\lambda}_{\bigstar_n})| = |\operatorname{spec}(H^{\mu}_{\bigstar_n})| = 1$ then $\lambda, \mu \in \{(n), (\frac{n}{2}, \frac{n}{2}), (\frac{n}{3}, \frac{n}{3}, \frac{n}{3})\}$. For these three cases, the star graph Hamiltonian has eigenvalue 0, $n + 2$ or $\frac{4}{3}n + 4$. Thus $\operatorname{spec}(H^{\lambda}_{\bigstar_n}) = \operatorname{spec}(H^{\mu}_{\bigstar_n})$ only if $\lambda = \mu$. ■

Remark 7.3. Let $n = 9$. For $\lambda = (3, 3, 3)$ and $\mu = (6, 2, 1)$ we have

$$\operatorname{spec}(H^{\lambda}_{\bigstar_n}) = \{16\} \subseteq \{20, 16, 6\} = \operatorname{spec}(H^{\mu}_{\bigstar_n}).$$

Therefore the role of K_n in Proposition 7.2(i) is essential (note that $\eta_\lambda = 72$ and $\eta_\mu = 48$). Likewise, the restriction to partitions with at most three rows is required (cf. Remark 6.14). Namely, let $n = 21$, $\lambda = (7, 7, 7)$ and $\mu = (9, 6, 5, 1)$. Then $\eta_\lambda = 336 = \eta_\mu$ and

$$\operatorname{spec}(H^{\lambda}_{\bigstar_n}) = \{16\} \subseteq \{12, 16, 18, 23\} = \operatorname{spec}(H^{\mu}_{\bigstar_n}).$$

Let $\lambda \vdash n$ be a three-row partition, and let m be the minimal polynomial of $H^{\lambda}_{\bigstar_n}$; note that m is of degree at most 3, and determined by Example 6.12. As a consequence of Proposition 7.2, the largest eigenvalue of H^{λ}_{G} for $d = 3$ is the solution of the NPO problem

$$\min \left\{ \alpha : \alpha - h_G = \sum_k s_k^* s_k + q \text{ for some } s_k \in \mathcal{F}_n,\ q \in \mathcal{I}_n^{\mathrm{Sw}_3} + \big(\eta_\lambda - h_{K_n}, m(h_{\bigstar_n})\big) \right\}.$$

7.2. **Separation of irreps via low-degree central elements.** In this section we show that partitions $\lambda \vdash n$ with at most d rows can be distinguished by d relations of degrees $1, \dots, d$, which can be used in an NPO problem for solving the localized d-QMC problem, i.e., finding the largest eigenvalue of H^{λ}_{G}.

For $2 \le k \le n$ let $q_k \in \mathbb{C}[S_n]$ be the sum of all k-cycles in S_n (there are $(k-1)!\binom{n}{k}$ of them). Since q_k is central in $\mathbb{C}[S_n]$, we have $\rho_\lambda(q_k) = \gamma_{k,\lambda} I$, where

$$\gamma_{k,\lambda}\, \chi_\lambda(e) = \operatorname{Tr}\big(\rho_\lambda(q_k)\big) = (k-1)! \binom{n}{k} \chi_\lambda((1\ \dots\ k)). \tag{7.3}$$

Note that

$$\gamma_{2,\lambda} = \eta_\lambda = n^2 + \frac{d(d-1)(2d-1)}{6} - \sum_{k=1}^{d} \big(\lambda_k - (k-1)\big)^2$$

by Proposition 6.7. The other values $\gamma_{k,\lambda}$ can be computed using the normalized character formula [Lsa08, Theorem 4] (with the Murnaghan-Nakayama rule, cf. Appendix C.1, at its core), and are in particular integers. For example,

$$\begin{aligned}
\gamma_{3,\lambda} &= \frac{1}{3} \cdot n(n-1)(n-2)\,\frac{\chi_\lambda((1\,2\,3))}{\chi_\lambda(e)} \\
&= \sum_{k=1}^{d}\sum_{j=1}^{\lambda_k}(j-k)^2 - \binom{n}{2} \\
&= -\binom{n}{2} + \sum_{k=1}^{d}\left(\frac{(\lambda_k-k)(\lambda_k-k+1)(2(\lambda_k-k)+1)}{6} + \frac{(k-1)k(2k-1)}{6}\right) \\
&= \frac{d(d-1)^2(d-2)}{2} - \binom{n}{2} + \sum_{k=1}^{d}\frac{(\lambda_k-k)(\lambda_k-k+1)(2(\lambda_k-k)+1)}{6}
\end{aligned}$$

using the formula after [Lsa08, Theorem 4]. The values $\gamma_{k,\lambda}$ separate irreps as follows.

Theorem 7.4. *If $\lambda, \mu \vdash n$ have at most d rows, then*

$$\lambda = \mu \qquad \Longleftrightarrow \qquad \gamma_{k,\lambda} = \gamma_{k,\mu} \text{ for all } k = 2, \dots, d.$$

Proof. Let $p_{d,k} = x_1^k + \cdots + x_d^k$ denote the kth power-sum symmetric polynomial in d variables. Let $\lambda \vdash n$ have at most d rows, and write $\lambda_i = 0$ for $\operatorname{ht}(\lambda) < i \le d$. By (7.3) we have

$$k\gamma_{k,\lambda} = n^{\downarrow k}\,\frac{\chi_\lambda((1\,\dots\,k))}{\chi_\lambda(e)},$$

where $n^{\downarrow k}$ is the falling factorial. By [VK81, Lemma 5.1] or [IO02, Propositions 1.4, 3.3 and 3.4],

$$k\gamma_{k,\lambda} = \Big(p_{d,k} + P_k\big(p_{d,1},\dots,p_{d,k-1}\big)\Big)\big(\lambda_1 - 1 + \tfrac{1}{2}, \lambda_2 - 2 + \tfrac{1}{2}, \dots, \lambda_d - d + \tfrac{1}{2}\big) \tag{7.4}$$

for some polynomial P_k in $k-1$ variables. Also note that

$$p_{d,1}\big(\lambda_1 - 1 + \tfrac{1}{2}, \lambda_2 - 2 + \tfrac{1}{2}, \dots, \lambda_d - d + \tfrac{1}{2}\big) = n - \tfrac{d^2}{2}.$$

Now assume that $\gamma_{k,\lambda} = \gamma_{k,\mu}$ for all $k = 2, \dots, d$. By (7.4),

$$p_{d,k}\big(\lambda_1 - 1 + \tfrac{1}{2}, \lambda_2 - 2 + \tfrac{1}{2}, \dots, \lambda_d - d + \tfrac{1}{2}\big) = p_{d,k}\big(\mu_1 - 1 + \tfrac{1}{2}, \mu_2 - 2 + \tfrac{1}{2}, \dots, \mu_d - d + \tfrac{1}{2}\big)$$

for all $k = 1, \dots, d$. Since

$$\lambda_1 - 1 + \tfrac{1}{2} > \lambda_2 - 2 + \tfrac{1}{2} > \cdots > \lambda_d - d + \tfrac{1}{2}, \qquad \mu_1 - 1 + \tfrac{1}{2} > \mu_2 - 2 + \tfrac{1}{2} > \cdots > \mu_d - d + \tfrac{1}{2}$$

and the power-sum symmetric polynomials distinguish points up to a coordinate shuffle, it follows that $\lambda = \mu$. ■

For $k \in \mathbb{N}$ denote

$$c_k = \sum_{\substack{1 \le i_0, \dots, i_k \le d \\ \text{pairwise distinct,} \\ i_0 < i_j \text{ for } j \ge 1}} \operatorname{swap}_{i_0 i_1}\operatorname{swap}_{i_0 i_2}\cdots\operatorname{swap}_{i_0 i_k} \in \mathcal{F}_n$$

which corresponds to $q_{k+1} \in \mathbb{C}[S_n]$. By Theorem 7.4, finding the largest eigenvalue of the localized d-QMC Hamiltonian H_G^λ (for $\mathrm{ht}(\lambda) \le d$) is equivalent to the NPO problem

$$\min \left\{ \alpha : \alpha - h_G = \sum_k s_k^* s_k + q \text{ for some } s_k \in \mathcal{F}_n,\ q \in \mathcal{I}_n^{\mathrm{Sw}_d} + \big(c_k - \gamma_{k+1,\lambda} \colon k \le d-1\big) \right\}.$$

As in Section 5, this NPO can be solved through a hierarchy of SDP relaxations.

References

[AMG20] A. Anshu, D. Gosset, K. Morenz, *Beyond Product State Approximations for a Quantum Analogue of Max Cut*, 15th Conference on the Theory of Quantum Computation, Communication and Cryptography (TQC 2020), Leibniz International Proceedings in Informatics 158, 7:1–7:15, Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2020.

[Aue94] A. Auerbach, *Interacting electrons and quantum magnetism*, Springer-Verlag, New York, 1994.

[BAMC09] K. S. D. Beach, F. Alet, M. Mambrini, S. Capponi, SU(N) *Heisenberg model on the square lattice: A continuous-N quantum Monte Carlo study*, Phys. Rev. B 80 (2009) 184401.

[BPT13] G. Blekherman, P.A. Parrilo, R.R. Thomas, Rekha R. (ed.), *Semidefinite optimization and convex algebraic geometry*, Society for Industrial and Applied Mathematics (SIAM), 2013.

[BDZ08] I. Bloch, J. Dalibard, W. Zwerger, *Many-body physics with ultracold gases*, Rev. Mod. Phys. 80 (2008) 885.

[BCEHK24] A. Bene Watts, A. Chowdhury, A. Epperly, J. W. Helton, I. Klep, *Relaxations and Exact Solutions to Quantum Max Cut via the Algebraic Structure of Swap Operators*, Quantum 8 (2024) 1352, 88pp.

[BCP97] W. Bosma, J. Cannon, C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. 24 (1997) 235–265.

[BGKT19] S. Bravyi, D. Gosset, R. König, K. Temme, *Approximation algorithms for quantum many-body problems*, J. Math. Phys. 60 (2019) 032203.

[BH13] F.G.S.L. Brandao, A. W. Harrow, *Product-state approximations to quantum ground states*, Proceedings of the Forty-Fifth Annual ACM Symposium on Theory of Computing (2013) 871–880.

[BKP16] S. Burgdorf, I. Klep, J. Povh, *Optimization of polynomials in non-commuting variables*, Springer, 2016.

[CJKKW+] C. Carlson, Z. Jorquera, A. Kolla, S. Kordonowy, S. Wayland, *Approximation Algorithms for Quantum Max-d-Cut*, preprint `arXiv:2309.10957`.

[DLTW08] A. C. Doherty, Y.-C. Liang, B. Toner, S. Wehner: *The quantum moment problem and bounds on entangled multi-prover games*, in Proceedings of the 2008 IEEE 23rd Annual Conference on Computational Complexity, 199–210, IEEE Computer Society, 2008.

[FJ97] A. Frieze, M. Jerrum, *Improved approximation algorithms for MAX k-CUT and MAX BISECTION*, Algorithmica 18 (1997), 67–81.

[Fro00] G. Frobenius, *Über die Charaktere der symmetrischen Gruppe*, Berl. Ber. (1900) 516–534.

[FH91] W. Fulton, J. Harris, *Representation theory. A first course*, Grad. Texts in Math., Read. Math. 129, Springer-Verlag, New York, 1991.

[GH24+] S. Gharibian, C. Hecht, *Hardness of approximation for ground state problems*, preprint `arXiv:2411.04874`.

[GK12] S. Gharibian, J. Kempe, *Approximation algorithms for QMA complete problems*, SIAM J. Comput. 41 (2012) 1028–1050.

[GP19] S. Gharibian, O. Parekh, *Almost Optimal Classical Approximation Algorithms for a Quantum Generalization of Max-Cut*, Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2019), Leibniz International Proceedings in Informatics 145, 31:1-31:17, Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2019.

[GW95] M. X. Goemans, D. P. Williamson, *Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming*, J. ACM, 42 (1995) 1115–1145.

[GNW21] D. Gross, S. Nezami, M. Walter, *Schur-Weyl duality for the Clifford group with applications: property testing, a robust Hudson theorem, and de Finetti representations*, Commun. Math. Phys. 385 (2021) 1325–1393.

[HM17] A. W. Harrow, A. Montanaro, *Extremal eigenvalues of local Hamiltonians*, Quantum 1 (2017) 6.

[HO22] M. B. Hastings, R. O'Donnell, *Optimizing strongly interacting fermionic Hamiltonians*, Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing (2022) 776–789.

[HKL20] D. Henrion, M. Korda, J.-B. Lasserre: *The moment-SOS hierarchy: Lectures in probability, statistics, computational geometry, control and nonlinear PDEs*, Optimization and Its Applications 4, World Scientific, 2020.

[HM04] J. W. Helton, S. A. McCullough, *A Positivstellensatz for non-commutative polynomials*, Trans. Amer. Math. Soc. 356 (2004) 3721–3737.

[HJ85] R. A. Horn, C. R. Johnson, *Matrix analysis*, Cambridge University Press, Cambridge, 1985.

[HTPG24+] F. Huber, K. Thompson, O. Parekh, S. Gharibian, *Second order cone relaxations for quantum Max Cut*, preprint `arXiv:2411.04120`.

[IO02] V. Ivanov, G. Olshanski, *Kerov's central limit theorem for the Plancherel measure on Young diagrams*, Symmetric Functions 2001: Surveys of Developments and Perspectives, NATO Science Series 74, Springer, Dordrecht, 2002.

[KT07] N. Kawashima, Y. Tanabe, *Ground States of the* $\mathrm{SU}(N)$ *Heisenberg Model*, Phys. Rev. Lett. 98 (2007) 057202.

[KKR06] J. Kempe, A. Yu. Kitaev, O. Regev, *The complexity of the local hamiltonian problem*, SIAM J. Comput. 35 (2006) 1070–1097.

[Kho02] S. Khot, *On the power of unique 2-prover 1-round games*, Proceedings of the thiry-fourth annual ACM symposium on Theory of computing (2002) 767–775.

[Kin23] R. King, *An Improved Approximation Algorithm for Quantum Max-Cut on Triangle-Free Graphs*, Quantum 7 (2023), 1180.

[KSV02] A. Yu. Kitaev, A. H. Shen, M. N. Vyalyi, *Classical and quantum computation*, Graduate studies in mathematics 47, American Mathematical Society, Providence, RI, 2002.

[LVV15] Z. Landau, U. Vazirani, T. Vidick, *A polynomial time algorithm for the ground state of one-dimensional gapped local Hamiltonians*, Nat. Phys. 11 (2015) 566–569.

[Lsa08] M. Lassalle, *An explicit formula for the characters of the symmetric group*, Math. Ann. 340 (2008) 383–405.

[Lse01] J.-B. Lasserre, *Global optimization with polynomials and the problem of moments*, SIAM J. Optim. 11 (2001) 796–817.

[Lau09] M. Laurent: *Sums of squares, moment matrices and optimization over polynomials*, in Emerging applications of algebraic geometry 157–270, IMA Vol. Math. Appl. 149, Springer, New York, 2009.

[Lee22] E. Lee, *Optimizing quantum circuit parameters via SDP*, 33rd International Symposium on Algorithms and Computation (ISAAC 2022), Leibniz International Proceedings in Informatics 248, 48:1-48:16, Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2022.

[LM16] F. Levkovich-Maslyuk, *The Bethe Ansatz*, J. Phys. A: Math. Theor. 49 (2016) 323004.

[LM62] E. Lieb, D. Mattis, *Ordering energy levels of interacting spin systems*, J. Math. Phys. 3 (1962) 749–751.

[MP14] B.D. McKay, A. Piperno, *Practical Graph Isomorphism, II*, J. Symbolic Comput. 60 (2014) 94–112.

[Mor86] F. Mora, *Groebner bases for noncommutative polynomial rings*, Algebraic algorithms and error correcting codes (Grenoble, 1985), 353–362, Lecture Notes in Comput. Sci. 229, Springer, Berlin, 1986.

[MPP18] A. Morales, I. Pak, G. Panova, *Hook formulas for skew shapes I. q-analogues and bijections*, J. Combin. Theory Ser. A 154 (2018), 350–405.

[Nar14] H. Naruse, *Schubert calculus and hook formula*, 73rd Sém. Lothar. Combin., Strobl, Austria, 2014. Talk slides available at `https://www.mat.univie.ac.at/ slc/wpapers/s73vortrag/naruse.pdf`.

[NPA08] M. Navascués, S. Pironio, A. Acín, *A convergent hierarchy of semidefinite programs characterizing the set of quantum correlations*, New J. Phys. 10 (2008) 073013.

[Nie23] J. Nie, *Moment and polynomial optimization*, Society for Industrial and Applied Mathematics (SIAM), 2023.

[NC10] M. A. Nielsen, I. L. Chuang, *Quantum computation and quantum information*, Cambridge University Press, Cambridge, 2000.

[PT21] O. Parekh, K. Thompson, *Application of the Level-2 Quantum Lasserre Hierarchy in Quantum Approximation Algorithms*, 48th International Colloquium on Automata, Languages, and Programming (ICALP 2021), Leibniz International Proceedings in Informatics 198, 102:1–102:20, Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2021.

[PT22] O. Parekh, K. Thompson, *An Optimal Product-State Approximation for 2-Local Quantum Hamiltonians with Positive Terms*, preprint `arXiv:2206.08342`.

[PM17] S. Piddock, A. Montanaro, *The complexity of antiferromagnetic interactions and 2D lattices*, Quantum Inf. Comput. 17 (2017) 636–672.

[PM21] S. Piddock, A. Montanaro, *Universal qudit Hamiltonians*, Commun. Math. Phys. 382 (2021) 721–771.

[PNA10] S. Pironio, M. Navascués, A. Acín, *Convergent relaxations of polynomial optimization problems with noncommuting variables*, SIAM J. Optim. 20 (2010) 2157–2180.

[Pro07] C. Procesi, *Lie groups. An approach through invariants and representations*, Universitext, Springer, New York, 2007.

[Pro21] C. Procesi, *A note on the Formanek Weingarten function*, Note Mat. 41 (2021) 69–109.

[Rag08] P. Raghavendra, *Optimal algorithms and inapproximability results for every CSP?*, Proceedings of the fortieth annual ACM symposium on Theory of computing (2008) 245–254.

[Rag09] P. Raghavendra, *Approximating NP-hard problems efficient algorithms and their limits*, University of Washington, 2009, PhD thesis.

[Sag01] B. E. Sagan, *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*, Graduate Texts in Mathematics 203, Springer, 2001.

[Sta99] R.P. Stanley, *Enumerative combinatorics*, Vol. 2, Cambridge University Press, Cambridge, 1999.

[TRZ+] J. Takahashi, C. Rayudu, C. Zhou, R. King, K. Thompson, O. Parekh, *An SU(2)-symmetric semidefinite programming hierarchy for Quantum Max Cut*, preprint `arXiv:2307.15688`

[VK81] A. M. Vershik, S. V. Kerov, *Asymptotic theory of the characters of a symmetric group*, Funktsional. Anal. i Prilozhen. 15 (1981) 15–27.

[WHSK20] Y. Wang, Z. Hu, B. C. Sanders, S. Kais, *Qudits and High-Dimensional Quantum Computing*, Front. Phys. 8 (2020) 589504.

Appendix A. Linear subspace of $M_{d^n}(\mathbb{C})$ spanned by the products of at most $d-1$ swap matrices

Here we prove that in $M_n^{\mathrm{Sw}_d}(\mathbb{C})$, there are no relations of order at most $d-1$ (in the swap matrices, represented by transpositions) other than (4.1).

Remark A.1. Note that if a permutation σ is a product of disjoint cycles of lengths $\ell_1, \ldots, \ell_k$ respectively, then σ is can be written as a product of $\sum_{i=1}^{k}(\ell_i - 1)$ transpositions (and not fewer than than many transpositions).

First we note that linear independence of swap operators is preserved if the local dimension increases.

Lemma A.2. *If a set of products of swap operators is linearly independent in $M_n^{\mathrm{Sw}_d}(\mathbb{C})$, then it is linearly independent in $M_n^{\mathrm{Sw}_{d+1}}(\mathbb{C})$.*

Proof. By Theorem 2.2, the algebra $M_n^{\mathrm{Sw}_d}(\mathbb{C})$ can be obtained as a quotient of $M_n^{\mathrm{Sw}_{d+1}}(\mathbb{C})$, namely by modding out the direct summands $\rho_\lambda(\mathbb{C}S_n)$, where λ is a partition of n with exactly d rows. The statement then follows since any linearly independent set in a quotient $M_n^{\mathrm{Sw}_{d+1}}(\mathbb{C})$ is linearly independent in $M_n^{\mathrm{Sw}_{d+1}}(\mathbb{C})$. ■

The following statement is the main result of this section.

Proposition A.3. *The set of all products (that correspond to distinct permutations) of at most $d-1$ swap matrices is a basis of the subspace of $M_n^{Sw_d}(\mathbb{C})$ of polynomials in the $Swap_{ij}$ of degree at most $d-1$.*

Remark A.4. In other words, Proposition A.3 states that permutations, which are products of at most $d-1$ transpositions, are linearly independent as elements of $M_n^{\mathrm{Sw}_d}(\mathbb{C})$. In Section 5, we mentioned another natural linearly independent subset of $M_n^{\mathrm{Sw}_d}(\mathbb{C})$. Recall that a permutation $\pi \in S_n$ is called $(d+1)$-*good* if there is no increasing sequence $j_0 < \cdots < j_d$ such that $\pi(j_0) > \cdots > \pi(j_d)$. Then $(d+1)$-good permutations form a basis of $M_n^{\mathrm{Sw}_d}(\mathbb{C})$ by [Pro21, Theorem 8]. However, Proposition A.3 is not a direct consequence of this result. Namely, a product of at most $d-1$ transpositions is not necessarily a $(d+1)$-good permutation if $d \geq 3$. Concretely, the product of $d-1$ disjoint transpositions $\pi = \prod_{i=1}^{d-1}(i, 2d-1-i)$ satisfies $\pi(1) > \pi(2) > \cdots > \pi(2d-2)$, so it is not $2(d-1)$-good (and in particular, not $(d+1)$-good if $d \geq 3$).

Before proving Proposition A.3, we require two lemmas.

Lemma A.5. *Let σ be a permutation in S_n that is a product of $d-1$ transpositions and cannot be written as a product of fewer than $d-1$ transpositions. Let $v \in (\mathbb{C}^d)^{\otimes n}$ be an elementary tensor whose factors are standard basis vectors (so that v has at most d distinct indices). Suppose that for any product τ of k disjoint cycles of σ, where $k = 1, 2$, the part of v on which τ acts has at most $k-1$ indices that are repeated and they occur at most twice. Then σ acts uniquely on v among the products of at most $d-1$ transpositions.*

Proof. Let σ and v be as in Lemma A.5. If σ is a permutation on strictly less than n letters, add to its cyclic structure the singletons corresponding to the missing letters in $\{1, \ldots, n\}$.

First suppose τ is one of the disjoint cycles of σ and let w be the part of v on which τ acts. If w has an index that appears at least twice, then one can construct at least one other permutation τ' that is a product of at most as many transpositions as τ, and gives the same result when applied to w. Indeed, to find the first cycle of τ', start with the index of w that is repeated, then find its image among the remaining indices, take the image of the latter and continue until the starting index occurs again. Since the starting index occurs at least twice in w, the produced cycle is of smaller length compared to σ. Now repeat the procedure starting with any other index to deduce what the other disjoint cycles are. This way we break the action of σ on w into an action of a product of disjoint cycles (some of them may have length one) whose lengths sum up to the length of σ. Hence their product can be written as a product of strictly less than $d-1$ transpositions.

Now let $\tau = \tau_1\tau_2$ be a product of two disjoint cycles of σ and let w_1 and w_2 be the parts of v on which τ_1 and τ_2 act, respectively. Suppose that none of w_1 and w_2 has repeated indices, but they do share at least two indices. For simplicity suppose they share exactly two indices. We want to construct another permutation τ' that is a product of at most as many transpositions as τ, and gives the same result when applied to $w_1 \otimes w_2$. As before, for the first cycle of τ' start with one of the indices with two occurrences in w_1, say j_1, then find its image among the remaining indices, take the image of the latter and continue until j_1 occurs again. Note that the first time when the image of a letter is the other index with two occurrences, call it j_2, there are two choices: to consider the image of j_2 by either τ_1 or τ_2 and then continue the process until the image of a letter is j_1 again. Note that if we choose to continue with $\tau_1(j_2)$, we get back τ, but if we consider $\tau_2(j_2)$, we switch to the other cycle and finish with the factor e_{j_1} of w_2. Hence, the second option produces the first cycle of the permutation τ' we are looking for. For the second cycle of τ' restart the procedure with the index j_2 of a factor in w_1. By construction, τ' is also a product of at most $d-1$ transpositions.

It remains to prove that if σ meets the conditions in Lemma A.5, then it acts on v uniquely among the products of at most $d-1$ transpositions. This is true by the same procedure as above of deducing the cyclic structure by comparing v to its image $\sigma(v)$. Indeed, start with any index, take its image and continue until the starting index occurs again.

The only time we have two options during this process is when we hit an index j in v that occurs (exactly) twice across two cycles, say τ_1 and τ_2. Denote the parts of v on which τ_i acts by w_i. Suppose we started the process with j in τ_1. When we hit j again, we can either terminate the process (which yields the cycle τ_1) or continue with the image of j by τ_2. In this case we join the two cycles τ_1 and τ_2, which means that the resulting permutation must have at least one transposition more than σ.

If we start with an index $k \neq j$ of τ_1, then, when we first hit j, there are again two options: either to continue with $\tau_1(j)$ or $\tau_2(j)$. The choice $\tau_1(j)$ at the end reproduces τ_1 (actually, it may happen that the index k occurs in another cycle, say τ_3, and we may switch to τ_3 after hitting k again, but this case was already treated before). With the choice $\tau_2(j)$ we switch to the other cycle τ_2 and since k does not occur in w_2, the process does not terminate in w_2 (meaning that we need to switch cycle once more before terminating). This again means that we join (at least) two cycles and the resulting permutation must have at least one transposition more than σ.

This shows that σ is the only permutation that can be written as a product of at most $d-1$ transpositions and gives the result $\sigma(v)$ when applied to v. ■

Example A.6. Let $d = 4, n = 5$ and define $\sigma = (12)(345)$. Let

$$v_1 = e_1 \otimes e_2 \otimes e_1 \otimes e_2 \otimes e_3 \quad \text{and} \quad v_2 = e_1 \otimes e_2 \otimes e_1 \otimes e_3 \otimes e_4.$$

It is easy to see that since σ has two disjoint cycles and v_1 has two indices that occur twice, σ acts on v_1 in the same way as $\sigma' = (14)(253)$. On the other hand, by the algorithm of deducing the cyclic structure by comparing a vector to its image, σ acts on v_2 uniquely among the products of at most 3 transpositions in S_5.

Lemma A.7. *Let σ in S_n be a product of $d-1$ transpositions that cannot be written as a product of fewer than $d-1$ transpositions. Then there is a vector $v \in (\mathbb{C}^d)^{\otimes n}$ whose tensor factors are standard basis vectors with at most d distinct indices that meets the conditions in Lemma A.5.*

Proof. Let $\sigma \in S_n$ be a product of $d-1$ transpositions which cannot be written as a product of less than $d-1$ transpositions. Suppose σ is a product of k disjoint cycles for some $k = 1, \dots, d-1$ (all the cycles being of length 2 or more). Hence, σ is a permutation on $d+k-1$ letters, but the factors of v are chosen among the d standard basis vectors of $\mathbb{C}^d$.

Without loss of generality assume that the letters of σ are $1, \dots, d+k-1$. Order the cycles of σ increasingly by their lengths. Then assign the indices i_j to the first $d+k-1$ factors e_{i_j} of v in the following way:

Assign indices $1, \dots, d$ (e.g., increasingly according to the position of the factors) to the part of v on which the cycles of σ involving letters $1, \dots, d$ act (the cycle with d may involve larger indices and hence we do not yet assign the indices to all of the factors of v on which this cycle acts). Now v has $k-1$ more factors to be labeled (with indices between 1 and d), hence the corresponding part of σ has at most $\lfloor (k-1)/2 \rfloor$ disjoint cycles. Since we have k cycles in total (each of length at least 2), the part of σ on the letters $1, \dots, d$ is a product of at least $\lfloor (k+1)/2 \rfloor$ disjoint cycles.

So assign to the next $\lfloor (k-1)/2 \rfloor$ unlabeled factors of v (e.g., increasingly according to the position of the factors) the first letters of the cycles of σ on the letters $1, \dots, d$. Finally, assign to the remaining unlabeled factors of v the second letters of the cycles of σ on the letters $1, \dots, d$.

This way we labeled the first $d+k-1$ factors of v. To the remaining factors just assign the index 1. It is now clear from the construction that the obtained vector meets the conditions in Lemma A.5. ■

Example A.8. Let us illustrate Lemmas A.5 and A.7 in the case $n = 8$ and $d = 5$. Suppose $\sigma \in S_8$ is a product of 4 transpositions and it cannot be written as a product of less than 4 transpositions (here we omit writing the singletons in σ). Then we have 4 options:

(a) If σ is a 5-cycle, e.g., $\sigma = (12345)$, then a suitable vector is

$$v = e_1 \otimes e_2 \otimes e_3 \otimes e_4 \otimes e_5 \otimes e_1 \otimes e_1 \otimes e_1.$$

(b) If σ has two cycles, there are two possible cyclic structures: two 3-cycles or a product of a transposition and a 4-cycle. E.g., $\sigma = (123)(456)$ or $\sigma = (12)(3456)$. In both cases we can take

$$v = e_1 \otimes e_2 \otimes e_3 \otimes e_4 \otimes e_5 \otimes e_1 \otimes e_1 \otimes e_1.$$

(c) If σ has 3 cycles, then it must be a product of two transpositions and a 3-cycle. E.g., if $\sigma = (12)(34)(567)$, we can take

$$v = e_1 \otimes e_2 \otimes e_3 \otimes e_4 \otimes e_5 \otimes e_1 \otimes e_3 \otimes e_1.$$

(d) If σ has 4 cycles, then it must be a product of four transpositions. If, e.g., $\sigma = (12)(34)(56)(78)$, we can take

$$v = e_1 \otimes e_2 \otimes e_3 \otimes e_4 \otimes e_5 \otimes e_1 \otimes e_2 \otimes e_4.$$

Proof of Proposition A.3. Denote the set of all products (that correspond to distinct permutations) of at most $d-1$ swap matrices by $\tilde{\mathcal{B}}_{d-1}$. Suppose

$$\sum_{s \in \tilde{\mathcal{B}}_{d-1}} \alpha_s s = 0$$

for some scalars α_s. By Lemma A.5 and Lemma A.7, for each product s of $d-1$ transpositions that cannot be written as a product of less than $d-1$ transpositions there is an elementary tensor vector $v_s \in (\mathbb{C}^d)^{\otimes n}$ such that s acts uniquely on v_s among the elements of $\tilde{\mathcal{B}}_{d-1}$. Since elements of $\tilde{\mathcal{B}}_{d-1}$ act on $(\mathbb{C}^d)^{\otimes n}$ as permutations of tensor factors, this means that $s \cdot v_s$ is linearly independent of $\{t \cdot v_s : t \in \tilde{\mathcal{B}}_{d-1} \setminus \{s\}\}$. Hence, $\alpha_s = 0$ for all $s \in \tilde{\mathcal{B}}_{d-1}$ that cannot be written as products of less than $d-1$ transpositions. Now use induction and Lemma A.2 to finish the proof. ∎

Appendix B. Swap matrices on $(\mathbb{C}^3)^{\otimes n}$ and $(\mathbb{C}^4)^{\otimes n}$

Here we give some results specific to the cases $d = 3$ and $d = 4$.

B.1. Linear space spanned by the products of at most two swap matrices.

We prove that in $M_n^{\mathrm{Sw}_3}(\mathbb{C})$, there are no relations of order two other than (4.1).

Proposition B.1. *The set $\mathcal{B}_2$ consisting of*

$$\begin{array}{rl} & I \\ \mathrm{Swap}_{ij} & i<j \\ \mathrm{Swap}_{ij}\mathrm{Swap}_{jk} & i<j<k \\ \mathrm{Swap}_{ij}\mathrm{Swap}_{ik} & i<j<k \\ \mathrm{Swap}_{ij}\mathrm{Swap}_{kl} & i<j,\ i<k<l \end{array}$$

is a basis of the subspace of $M_n^{Sw_3}(\mathbb{C})$ of polynomials in the $Swap_{ij}$ of degree at most two.

Proof. To prove the linear independence of $\mathcal{B}_2$ suppose

$$\begin{aligned} aI + \sum_{i<j} b_{ij}\,\mathrm{Swap}_{ij} + \sum_{i<j<k} c_{ijk}\,\mathrm{Swap}_{ij}\mathrm{Swap}_{jk} + \sum_{i<j<k} d_{ijk}\,\mathrm{Swap}_{ij}\mathrm{Swap}_{ik} + \\ \sum_{\substack{i<j\\ i<k<l}} e_{ijkl}\,\mathrm{Swap}_{ij}\mathrm{Swap}_{kl} = 0. \end{aligned} \tag{B.1}$$

for some scalars $a, b_{ij}, c_{ijk}, d_{ijk}, e_{ijkl}$.

To prove that the e_{ijkl} must all be zero, first consider the vector

$$v = e_1 \otimes e_2 \otimes e_3 \otimes e_1 \otimes e_1 \cdots \otimes e_1 \in \left(\mathbb{C}^d\right)^{\otimes n}.$$

Evaluate (B.1) on v to see that the term $\mathrm{Swap}_{1,2}\mathrm{Swap}_{3,4}$ is the only one that yields $e_2 \otimes e_1 \otimes e_1 \otimes e_3 \otimes e_1 \cdots \otimes e_1$. Hence, e_{1234} must be zero and by analogy, all of the e_{ijkl} must be zero as well.

A similar argument allows us to get rid of the c_{ijk} and the d_{ijk}. Indeed, after evaluating (B.1) on $v = e_1 \otimes e_2 \otimes e_3 \otimes e_1 \otimes e_1 \cdots \otimes e_1$, the term $\mathrm{Swap}_{1,2}\mathrm{Swap}_{2,3}$ is the only one that gives $e_3 \otimes e_1 \otimes e_2 \otimes e_1 \otimes e_1 \cdots \otimes e_1$ and $\mathrm{Swap}_{1,2}\mathrm{Swap}_{1,3}$ is the only one that gives $e_2 \otimes e_3 \otimes e_1 \otimes e_1 \otimes e_1 \cdots \otimes e_1$.

Finally, we are left with a linear combination of single swap matrices and the identity, which are clearly linearly independent. ∎

B.2. **Gell-Mann matrices of size** 3×3. Recall the definition of the Gell-Mann matrices from Subsection 1.5.1. For $d = 3$, there are eight Gell-Mann matrices, namely

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

They are self-adjoint, have trace zero and together with the identity $\lambda_0 := I$, they form a basis for $M_3(\mathbb{C})$. They satisfy

$$\lambda_a\lambda_b = \frac{2}{3}\,\delta_{a,b}\, I + \sum_{c=1}^{8} (d^{a,b,c} + \mathrm{i} f^{a,b,c})\,\lambda_c, \tag{B.2}$$

where $\delta_{a,b}$ is the Kronecker delta and the $f^{a,b,c}$ and $d^{a,b,c}$ are structure constants with

$$f^{a,b,c} = -\frac{1}{4}\mathrm{i}\,\mathrm{tr}(\lambda_a[\lambda_b,\lambda_c]) \quad \text{and} \quad d^{a,b,c} = \frac{1}{4}\,\mathrm{tr}(\lambda_a\{\lambda_b,\lambda_c\}).$$

Here $[A,B] = AB - BA$ and $\{A,B\} = AB + BA$ denote the commutator and the anticommutator respectively. Note that the $f^{a,b,c}$ are antisymmetric and the $d^{a,b,c}$ are symmetric under the interchange of any pair of indices. The nonzero $f^{a,b,c}$ are

$$f^{1,2,3} = 1, \quad f^{1,4,7} = f^{1,6,5} = f^{2,4,6} = f^{2,5,7} = f^{3,4,5} = f^{3,7,6} = \frac{1}{2}, \quad f^{4,5,8} = f^{6,7,8} = \frac{\sqrt{3}}{2},$$

while the nonzero $d^{a,b,c}$ are

$$d^{1,4,6} = d^{1,5,7} = d^{2,5,6} = d^{3,4,4} = d^{3,5,5} = \frac{1}{2}, \quad d^{2,4,7} = d^{3,6,6} = d^{3,7,7} = -\frac{1}{2},$$
$$d^{1,1,8} = d^{2,2,8} = d^{3,3,8} = \frac{1}{\sqrt{3}}, \quad d^{8,8,8} = -\frac{1}{\sqrt{3}},$$
$$d^{4,4,8} = d^{5,5,8} = d^{6,6,8} = d^{7,7,8} = -\frac{1}{2\sqrt{3}}.$$

Fix $n \in \mathbb{N}$. As in Section 1.5, denote

$$\lambda_a^j := \underbrace{I \otimes \cdots \otimes I}_{j-1} \otimes \lambda_a \otimes I \otimes \cdots \otimes I \in M_{3^n}(\mathbb{C})$$

for $a \in \{0,\ldots,8\}$. Then,

$$\{\lambda^1_{a_1}\lambda^2_{a_2}\cdots\lambda^n_{a_n} \mid a_j \in \{0,\ldots,8\},\ j = 1,\ldots,n\} \tag{B.3}$$

is a basis of $M_{3^n}(\mathbb{C})$, and $\lambda^i_{a_i}$ and $\lambda^j_{a_j}$ commute for $i \neq j$. By Proposition 1.9, each qutrit swap matrix can be written as a linear combination of the Gell-Mann matrices as follows:

$$\text{Swap}_{ij} = \frac{1}{3}I + \frac{1}{2}\sum_{a=1}^{8} \lambda^i_a \lambda^j_a. \tag{B.4}$$

B.3. **Linear subspace of $M_{3^n}(\mathbb{C})$ spanned by the products of at most three swap matrices.** Throughout, any two tuples (i, j) and (k, l) are compared w.r.t. the lex ordering.

Proposition B.2. *The set $\mathcal{B}_3$ consisting of $\mathcal{B}_2$ and the three types of cubics*

$$\text{Swap}_{ij}\text{Swap}_{kl}\text{Swap}_{pq} \quad i < j,\ k < l,\ p < q,\ (i, j) < (k, l) < (p, q); \tag{B.5}$$

$$\begin{aligned} &\text{Swap}_{ij}\text{Swap}_{jk}\text{Swap}_{pq} \quad i < j < k,\ p < q,\ p, q \notin \{i, j, k\},\\ &\text{Swap}_{ij}\text{Swap}_{ik}\text{Swap}_{pq} \quad i < j < k,\ p < q,\ p, q \notin \{i, j, k\}; \end{aligned} \tag{B.6}$$

$$\begin{aligned} &\text{Swap}_{ij}\text{Swap}_{jk}\text{Swap}_{kl},\ \text{Swap}_{ij}\text{Swap}_{jl}\text{Swap}_{kl},\ \text{Swap}_{ik}\text{Swap}_{jk}\text{Swap}_{jl},\\ &\text{Swap}_{ik}\text{Swap}_{kl}\text{Swap}_{jl},\ \text{Swap}_{il}\text{Swap}_{jl}\text{Swap}_{jk} \quad i < j < k < l; \end{aligned} \tag{B.7}$$

is a basis of the subspace of $M_n^{Sw_3}(\mathbb{C})$ of polynomials in the Swap_{ij} of degree at most three.

Remark B.3. As it can be seen from the proof, any of the cubics in (B.7) can be replaced by Swap_{il} Swap_{kl} Swap_{jk}.

Proof. For the spanning property of $\mathcal{B}_3$, first note that by Proposition B.1, every product of three swap matrices involving at least five indices is in the linear span of $\mathcal{B}_3$ and by (3.4), every product of three swap matrices involving four indices is in the linear span of $\mathcal{B}_3$ as well. This is because a product of three swap matrices involving five (resp. six) indices corresponds to a product of a 3-cycle and a disjoint transposition (resp. a product of three disjoint transpositions; these are in the span of $\mathcal{B}_2$). Similarly, a product of three swap matrices involving four indices corresponds to either a 4-cycle or a product of two disjoint transpositions (the latter being in the span of $\mathcal{B}_2$). Moreover, any product of three swap matrices involving three indices or less clearly corresponds to an element in $\mathcal{B}_2$ (either to a 3-cycle, a transposition or to the identity). This proves the spanning property of $\mathcal{B}_3$.

The proof of the linear independence of $\mathcal{B}_3$ relies heavily on the properties of the Gell-Mann matrices presented in Subsection B.2. Suppose there is a linear dependence among the elements of $\mathcal{B}_3$. Then, using (B.4), express each of the appearing terms w.r.t. the basis (B.3) consisting of different combinations of tensor products of the eight Gell-Mann matrices.

First, consider the elements in (B.5) and observe that for any choice of $i < j, k < l, p < q$ with $(i, j) < (k, l) < (p, q)$, the highest order terms in the expansion of $\text{Swap}_{ij}\text{Swap}_{kl}\text{Swap}_{pq}$ are of the form

$$\lambda^i_a \lambda^j_a \lambda^k_b \lambda^l_b \lambda^p_c \lambda^q_c, \qquad a, b, c \in \{1, \dots, 8\}.$$

Likewise, considering the elements in (B.6), for any choice of $i < j < k, p < q$ with $p, q \notin \{i, j, k\}$, the highest order terms in the expansion of $\text{Swap}_{ij}\text{Swap}_{jk}\text{Swap}_{pq}$ are of the form

$$\lambda^i_a \lambda^j_a \lambda^j_b \lambda^k_b \lambda^p_c \lambda^q_c, \qquad a, b, c \in \{1, \dots, 8\},$$

while for $\text{Swap}_{ij}\text{Swap}_{ik}\text{Swap}_{pq}$ they are of the form

$$\lambda^i_a \lambda^j_a \lambda^i_b \lambda^k_b \lambda^p_c \lambda^q_c = \lambda^j_a \lambda^i_a \lambda^i_b \lambda^k_b \lambda^p_c \lambda^q_c, \qquad a, b, c \in \{1, \dots, 8\}.$$

As for the elements in (B.7), for any choice of $i < j < k < l$, the highest order terms e.g. in the expansion of $\text{Swap}_{ij}\text{Swap}_{jk}\text{Swap}_{kl}$ are of the form

$$\lambda_a^i\,\lambda_a^j\,\lambda_b^j\,\lambda_b^k\,\lambda_c^k\,\lambda_c^l, \qquad a,b,c \in \{1,\dots,8\}$$

and similarly for the other four cases in (B.7).

We now gradually eliminate the terms in the linear dependence equation: (a) By the product formula (B.2), the elements in (B.5) are the only ones that have terms of order six and more precisely, for any choice of $i < j, k < l, p < q$ with $(i,j) < (k,l) < (p,q)$, the element $\text{Swap}_{ij}\text{Swap}_{kl}\text{Swap}_{pq}$ has the term $\lambda_1^i\,\lambda_1^j\,\lambda_2^k\,\lambda_2^l\,\lambda_3^p\,\lambda_3^q$, which does not appear in the expansion of any other element of $\mathcal{B}_3$. Hence, the coefficients next to each of the elements in (B.5) have to be zero.

(b) Now the elements in (B.6) are the only ones that have terms of order five. By (B.2),

$$\begin{aligned}
\lambda_1\lambda_3 &= \mathfrak{i}f^{1,3,2}\,\lambda_2 = -\mathfrak{i}f^{1,2,3}\,\lambda_2 = -\mathfrak{i}\,\lambda_2,\\
\lambda_2\lambda_3 &= \mathfrak{i}f^{2,3,1}\lambda_1 = \mathfrak{i}f^{1,2,3}\,\lambda_1 = \mathfrak{i}\,\lambda_1,\\
\lambda_1\lambda_6 &= d^{1,6,4}\lambda_4 = d^{1,4,6}\lambda_4 = \frac{1}{2}\lambda_4.
\end{aligned}$$

Hence, for any choice of $i < j < k, p < q$ with $p, q \notin \{i,j,k\}$, the element $\text{Swap}_{ij}\text{Swap}_{jk}\text{Swap}_{pq}$ has in its expansion

$$\lambda_2^i\,\lambda_2^j\,\lambda_3^j\,\lambda_3^k\,\lambda_5^p\,\lambda_5^q + \lambda_1^i\,\lambda_1^j\,\lambda_6^j\,\lambda_6^k\,\lambda_5^p\,\lambda_5^q = \mathfrak{i}\,\lambda_2^i\,\lambda_1^j\,\lambda_3^k\,\lambda_5^p\,\lambda_5^q + \frac{1}{2}\lambda_1^i\,\lambda_4^j\,\lambda_6^k\,\lambda_5^p\,\lambda_5^q.$$

But $\text{Swap}_{ij}\text{Swap}_{ik}\text{Swap}_{pq}$ has in its expansion

$$\lambda_1^i\,\lambda_1^j\,\lambda_3^i\,\lambda_3^k\,\lambda_5^p\,\lambda_5^q + \lambda_1^i\,\lambda_1^j\,\lambda_6^i\,\lambda_6^k\,\lambda_5^p\,\lambda_5^q = -\mathfrak{i}\,\lambda_2^i\,\lambda_1^j\,\lambda_3^k\,\lambda_5^p\,\lambda_5^q + \frac{1}{2}\lambda_1^i\,\lambda_4^j\,\lambda_6^k\,\lambda_5^p\,\lambda_5^q.$$

Since the quotient of any two coefficients next to the same basis element in the expansion of $\text{Swap}_{ij}\text{Swap}_{jk}\text{Swap}_{pq}$ and $\text{Swap}_{ij}\text{Swap}_{ik}\text{Swap}_{pq}$ must be the same, the above implies that the coefficient next to each of the elements in (B.6) has to be zero.

(c) So the elements in (B.7) are now the only ones in the linear dependence equation that have terms of order four and for any choice of $i < j < k < l$, only the five products listed in (B.7) have basis elements of the form $\lambda_a^i\lambda_b^j\lambda_c^k\lambda_d^l$. Denote the coefficients in the linear dependence equation before the products in (B.7) by $\alpha_1, \alpha_2, \dots, \alpha_5$ respectively.

We now consider the equations that we get by reading off the coefficients next to the basis elements $\lambda_a^i\lambda_b^j\lambda_c^k\lambda_d^l$ for several choices of a,b,c,d with a,b,c,d being all different numbers. First one can compute the following part of the expansion of $\text{Swap}_{ij}\text{Swap}_{jk}\text{Swap}_{kl}$,

$$-\frac{1}{2}\,\lambda_3^i\lambda_1^j\lambda_4^k\lambda_6^l + \frac{1}{2}\,\lambda_3^i\lambda_1^j\lambda_6^k\lambda_4^l + \frac{1}{2}\,\lambda_3^i\lambda_4^j\lambda_6^k\lambda_1^l - \frac{1}{2}\,\lambda_3^i\lambda_6^j\lambda_4^k\lambda_1^l.$$

Note that by permuting i,j,k,l, we can obtain four terms in the expansion of the other four elements in (B.7). E.g., by interchanging k and l, we see that $\text{Swap}_{ij}\text{Swap}_{jl}\text{Swap}_{kl}$ has in its expansion the four terms

$$-\frac{1}{2}\,\lambda_3^i\lambda_1^j\lambda_6^k\lambda_4^l + \frac{1}{2}\,\lambda_3^i\lambda_1^j\lambda_4^k\lambda_6^l + \frac{1}{2}\,\lambda_3^i\lambda_4^j\lambda_1^k\lambda_6^l - \frac{1}{2}\,\lambda_3^i\lambda_6^j\lambda_1^k\lambda_4^l.$$

Using this, one can easily obtain the following equations by comparing the coefficients next to several terms of the form $\lambda_a^i\lambda_b^j\lambda_c^k\lambda_d^l$:

$$\lambda_3^i\lambda_1^j\lambda_6^k\lambda_4^l: \quad -\alpha_1 + \alpha_2 + \alpha_4 = 0$$

$$\begin{aligned}
\lambda_3^i\lambda_6^j\lambda_4^k\lambda_1^l: &\quad -\alpha_1+\alpha_3+\alpha_5=0\\
\lambda_3^i\lambda_4^j\lambda_1^k\lambda_6^l: &\quad \alpha_2-\alpha_3+\alpha_4+\alpha_5=0\\
\lambda_4^i\lambda_1^j\lambda_3^k\lambda_6^l: &\quad -\alpha_1+\alpha_3+\alpha_4=0\\
\lambda_4^i\lambda_6^j\lambda_1^k\lambda_3^l: &\quad \alpha_1-\alpha_2+\alpha_5=0\\
\lambda_4^i\lambda_1^j\lambda_6^k\lambda_3^l: &\quad -\alpha_2+\alpha_3+\alpha_4-\alpha_5=0\\
\lambda_5^i\lambda_2^j\lambda_3^k\lambda_7^l: &\quad -\alpha_1+\alpha_3-\alpha_4=0.
\end{aligned}$$

The above system of equations has a unique solution $\alpha_1=\alpha_2=\cdots=\alpha_5=0$. This proves that for any choice of $i<j<k<l$, the coefficients in the linear dependence equation before the elements in (B.7) are zero.

(d) We are left with a linear dependence involving terms of degree at most two, which contradicts linear independence of $\mathcal{B}_2$ as shown in Proposition B.1. ■

B.4. **Linear subspace of $M_{3^n}(\mathbb{C})$ spanned by the products of at most four swap matrices.** Recall from the previous subsection that we compare tuples (i,j) and (k,l) w.r.t. the lex ordering.

Proposition B.4. *The set $\hat{\mathcal{B}}_4$ consisting of $\mathcal{B}_3$ and the following quartics*

$$\text{(B.8)}\qquad \mathrm{Swap}_{ij}\mathrm{Swap}_{kl}\mathrm{Swap}_{pq}\mathrm{Swap}_{rs} \qquad i<j,\ k<l,\ p<q,\ r<s,\ (i,j)<(k,l)<(p,q)<(r,s);$$

$$\text{(B.9)}\qquad \begin{array}{ll}\mathrm{Swap}_{ij}\mathrm{Swap}_{jk}\mathrm{Swap}_{pq}\mathrm{Swap}_{rs} & i<j<k,\ p<q,r<s,\\ \mathrm{Swap}_{ij}\mathrm{Swap}_{ik}\mathrm{Swap}_{pq}\mathrm{Swap}_{rs} & p,q,r,s\notin\{i,j,k\},(p,q)<(r,s);\end{array}$$

$$\text{(B.10)}\qquad \begin{array}{l}\mathrm{Swap}_{ij}\mathrm{Swap}_{jk}\mathrm{Swap}_{kl}\mathrm{Swap}_{pq},\ \mathrm{Swap}_{ij}\mathrm{Swap}_{jl}\mathrm{Swap}_{kl}\mathrm{Swap}_{pq},\\ \mathrm{Swap}_{ik}\mathrm{Swap}_{jk}\mathrm{Swap}_{jl}\mathrm{Swap}_{pq},\ \mathrm{Swap}_{ik}\mathrm{Swap}_{kl}\mathrm{Swap}_{jl}\mathrm{Swap}_{pq},\\ \mathrm{Swap}_{il}\mathrm{Swap}_{jl}\mathrm{Swap}_{jk}\mathrm{Swap}_{pq}\quad i<j<k<l,\ p<q,\ p,q\notin\{i,j,k,l\};\end{array}$$

$$\text{(B.11)}\qquad \begin{array}{l}\mathrm{Swap}_{ij}\mathrm{Swap}_{jk}\mathrm{Swap}_{pq}\mathrm{Swap}_{qr},\ \mathrm{Swap}_{ij}\mathrm{Swap}_{jk}\mathrm{Swap}_{pq}\mathrm{Swap}_{pr},\\ \mathrm{Swap}_{ij}\mathrm{Swap}_{ik}\mathrm{Swap}_{pq}\mathrm{Swap}_{pr},\quad i<j<k,\ p<q<r,\ i<p,\\ \qquad\{i,j,k\}\cap\{p,q,r\}=\emptyset;\end{array}$$

$$\text{(B.12)}\qquad \begin{array}{l}\mathrm{Swap}_{ij}\mathrm{Swap}_{ik}\mathrm{Swap}_{jl}\mathrm{Swap}_{jm},\ \mathrm{Swap}_{ij}\mathrm{Swap}_{ik}\mathrm{Swap}_{jl}\mathrm{Swap}_{km},\\ \mathrm{Swap}_{ij}\mathrm{Swap}_{ik}\mathrm{Swap}_{kl}\mathrm{Swap}_{km},\ \mathrm{Swap}_{ij}\mathrm{Swap}_{il}\mathrm{Swap}_{jk}\mathrm{Swap}_{jm},\\ \mathrm{Swap}_{ij}\mathrm{Swap}_{im}\mathrm{Swap}_{jk}\mathrm{Swap}_{jl},\ \mathrm{Swap}_{ij}\mathrm{Swap}_{il}\mathrm{Swap}_{im}\mathrm{Swap}_{jk},\\ \mathrm{Swap}_{ij}\mathrm{Swap}_{ik}\mathrm{Swap}_{im}\mathrm{Swap}_{jl},\ \mathrm{Swap}_{ij}\mathrm{Swap}_{ik}\mathrm{Swap}_{im}\mathrm{Swap}_{kl},\\ \mathrm{Swap}_{ij}\mathrm{Swap}_{ik}\mathrm{Swap}_{il}\mathrm{Swap}_{im},\ \mathrm{Swap}_{ij}\mathrm{Swap}_{ik}\mathrm{Swap}_{il}\mathrm{Swap}_{lm},\\ \mathrm{Swap}_{ij}\mathrm{Swap}_{ik}\mathrm{Swap}_{il}\mathrm{Swap}_{km},\ \mathrm{Swap}_{ij}\mathrm{Swap}_{ik}\mathrm{Swap}_{il}\mathrm{Swap}_{jm},\\ \qquad i<j<k<l<m,\end{array}$$

is a basis of the subspace of $M_n^{Sw_3}(\mathbb{C})$ of polynomials of degree at most four in the $Swap_{ij}$.

Proof. The spanning property of $\hat{\mathcal{B}}_4$ follows after identifying the products of swap matrices with permutations in S_n using the degree-reducing relation (1.5) with $d=3$. Indeed, considering the elements which correspond to the product of a 4-cycle and a disjoint transposition, the type $\mathrm{Swap}_{il}\mathrm{Swap}_{kl}\mathrm{Swap}_{jk}\mathrm{Swap}_{pq}$ missing in (B.10) is clearly in the span of $\hat{\mathcal{B}}_4$ by (1.5). As for the elements that correspond to 5-cycles, there are 12 of the total 24 5-cycles on the letters i, j, k, l, m missing in (B.12). Their expansions in terms of the elements of $\hat{\mathcal{B}}_4$ are given in Subsection B.4.1.

Now suppose there is a linear dependence between the elements of $\hat{\mathcal{B}}_4$ and express the appearing terms w.r.t. the basis (B.3) using the formula (B.4). We gradually eliminate terms from this relation starting with the ones with highest order terms.

(a) Consider the elements in (B.8). For any choice of indices $i < j, k < l, p < q, r < s$ with $(i, j) < (k, l) < (p, q) < (r, s)$, the highest order terms in the expansion of $\mathrm{Swap}_{ij}\mathrm{Swap}_{kl}\mathrm{Swap}_{pq}\mathrm{Swap}_{rs}$ are of the form

$$\lambda_a^i\,\lambda_a^j\,\lambda_b^k\,\lambda_b^l\,\lambda_c^p\,\lambda_c^q\,\lambda_d^r\,\lambda_d^s, \qquad a, b, c, d \in \{1, \dots, 8\}.$$

The product formula (B.2) implies that the elements in (B.8) are the only ones in $\hat{\mathcal{B}}_4$ with such terms and more precisely, for any choice of $i < j, k < l, p < q, r < s$ with $(i, j) < (k, l) < (p, q) < (r, s)$, the element $\mathrm{Swap}_{ij}\mathrm{Swap}_{kl}\mathrm{Swap}_{pq}\mathrm{Swap}_{rs}$ has the term $\lambda_1^i\,\lambda_1^j\,\lambda_2^k\,\lambda_2^l\,\lambda_3^p\,\lambda_3^q\lambda_4^r\,\lambda_4^s$, which does not appear in the expansion of any other element of $\hat{\mathcal{B}}_4$. Hence, by analogy, the coefficients next to each of the elements in (B.8) are zero.

(b) Now the elements in (B.9) are the only ones with highest order terms of degree 7, meaning, involving 7 distinct indices. Using part (b) of the proof of Proposition B.2, for any choice of $i < j < k, p < q, r < s$ with $p, q, r, s \notin \{i, j, k\}, (p, q) < (r, s)$, the element $\mathrm{Swap}_{ij}\mathrm{Swap}_{jk}\mathrm{Swap}_{pq}\mathrm{Swap}_{rs}$ has in its expansion

$$\lambda_2^i\,\lambda_2^j\,\lambda_3^j\,\lambda_3^k\,\lambda_5^p\,\lambda_5^q\,\lambda_9^r\,\lambda_9^s + \lambda_1^i\,\lambda_1^j\,\lambda_6^j\,\lambda_6^k\,\lambda_5^p\,\lambda_5^q\,\lambda_9^r\,\lambda_9^s = \mathfrak{i}\,\lambda_2^i\,\lambda_1^j\,\lambda_3^k\,\lambda_5^p\,\lambda_5^q\,\lambda_9^r\,\lambda_9^s + \frac{1}{2}\lambda_1^i\,\lambda_4^j\,\lambda_6^k\,\lambda_5^p\,\lambda_5^q\,\lambda_9^r\,\lambda_9^s.$$

But $\mathrm{Swap}_{ij}\mathrm{Swap}_{ik}\mathrm{Swap}_{pq}$ has in its expansion

$$\lambda_1^i\,\lambda_1^j\,\lambda_3^i\,\lambda_3^k\,\lambda_5^p\,\lambda_5^q\,\lambda_9^r\,\lambda_9^s + \lambda_1^i\,\lambda_1^j\,\lambda_6^i\,\lambda_6^k\,\lambda_5^p\,\lambda_5^q\,\lambda_9^r\,\lambda_9^s = -\mathfrak{i}\,\lambda_2^i\,\lambda_1^j\,\lambda_3^k\,\lambda_5^p\,\lambda_5^q\,\lambda_9^r\,\lambda_9^s + \frac{1}{2}\lambda_1^i\,\lambda_4^j\,\lambda_6^k\,\lambda_5^p\,\lambda_5^q\,\lambda_9^r\,\lambda_9^s.$$

By the same argument as in part (b) of the proof of Proposition B.2, all the coefficients next to the elements in (B.9) are zero.

(c) The elements in (B.10) and (B.11) are now the only ones with with highest order terms of degree 6. We first consider those in (B.11). For fixed $i < j < k, p < q < r, i < p$ with $\{i, j, k\} \cap \{p, q, r\} = \emptyset$, denote the coefficients next to the elements

$$\mathrm{Swap}_{ij}\mathrm{Swap}_{jk}\mathrm{Swap}_{pq}\mathrm{Swap}_{qr}, \mathrm{Swap}_{ij}\mathrm{Swap}_{jk}\mathrm{Swap}_{pq}\mathrm{Swap}_{pr}, \mathrm{Swap}_{ij}\mathrm{Swap}_{ik}\mathrm{Swap}_{pq}\mathrm{Swap}_{pr}$$

by β_1, β_2 and β_3 respectively. Clearly, these are the only elements in (B.11) whose highest order terms involve precisely the positions i, j, k, p, q, r. So comparing the coefficients next to the basis elements $\lambda_2^i\,\lambda_4^j\,\lambda_6^k\,\lambda_1^p\,\lambda_5^q\,\lambda_7^r$, $\lambda_2^i\,\lambda_5^j\,\lambda_6^k\,\lambda_1^p\,\lambda_4^q\,\lambda_7^r$ and $\lambda_1^i\,\lambda_2^j\,\lambda_3^k\,\lambda_4^p\,\lambda_5^q\,\lambda_8^r$ give the following equations

$$\begin{aligned}
\lambda_2^i\,\lambda_4^j\,\lambda_6^k\,\lambda_1^p\,\lambda_5^q\,\lambda_7^r: &\quad -\mathfrak{i}\frac{1}{4}\beta_1 - \mathfrak{i}\frac{1}{4}\beta_2 + \mathfrak{i}\frac{1}{4}\beta_3 = 0,\\
\lambda_2^i\,\lambda_5^j\,\lambda_6^k\,\lambda_1^p\,\lambda_4^q\,\lambda_7^r: &\quad -\mathfrak{i}\frac{1}{4}\beta_1 + \mathfrak{i}\frac{1}{4}\beta_2 + \mathfrak{i}\frac{1}{4}\beta_3 = 0,\\
\lambda_1^i\,\lambda_2^j\,\lambda_3^k\,\lambda_4^p\,\lambda_5^q\,\lambda_8^r: &\quad -\frac{\sqrt{3}}{2}\beta_1 + \frac{\sqrt{3}}{2}\beta_2 - \frac{\sqrt{3}}{2}\beta_3 = 0.
\end{aligned}$$

The above system has a unique solution $\beta_1 = \beta_2 = \beta_3 = 0$. Note that each of the highest order terms of the elements in (B.10) necessarily has one of the Gell-Mann matrices λ repeated twice. So the coefficients next to the basis elements $\lambda_2^i\,\lambda_4^j\,\lambda_6^k\,\lambda_1^p\,\lambda_5^q\,\lambda_7^r$, $\lambda_2^i\,\lambda_5^j\,\lambda_6^k\,\lambda_1^p\,\lambda_4^q\,\lambda_7^r$

and $\lambda_1^i\,\lambda_2^j\,\lambda_3^k\,\lambda_4^p\,\lambda_5^q\,\lambda_8^r$ in the expansion of the elements in (B.10) are zero. Similarly, each of the highest order terms of the elements corresponding to products of three disjoint transpositions has (at most) three distinct Gell-Mann matrices, each repeated twice. Hence, the coefficients next to the basis elements $\lambda_2^i\,\lambda_4^j\,\lambda_6^k\,\lambda_1^p\,\lambda_5^q\,\lambda_7^r$, $\lambda_2^i\,\lambda_5^j\,\lambda_6^k\,\lambda_1^p\,\lambda_4^q\,\lambda_7^r$ and $\lambda_1^i\,\lambda_2^j\,\lambda_3^k\,\lambda_4^p\,\lambda_5^q\,\lambda_8^r$ in the expansions of those elements are zero as well. We conclude that the coefficients next to the elements

$$\mathrm{Swap}_{ij}\mathrm{Swap}_{jk}\mathrm{Swap}_{pq}\mathrm{Swap}_{qr},\ \mathrm{Swap}_{ij}\mathrm{Swap}_{jk}\mathrm{Swap}_{pq}\mathrm{Swap}_{pr},\ \mathrm{Swap}_{ij}\mathrm{Swap}_{ik}\mathrm{Swap}_{pq}\mathrm{Swap}_{pr},$$

are zero and by analogy, the coefficients next to all the elements in (B.11) are zero.

Having eliminated the elements in (B.11), the fact that the coefficients next to the elements in (B.10) are zero easily follows from part (c) of the proof of Proposition B.2.

(d) Now the elements in (B.12) are the only ones in $\hat{\mathcal{B}}_4$ with highest order terms of degree 5. For fixed $i<j<k<l<m$ denote the coefficients next to the quartics in (B.12) by $\gamma_1,\dots,\gamma_{12}$ respectively and note that these are the only elements in (B.12) whose highest order terms involve precisely the positions i,j,k,l,m. Similar to before, we now compare the coefficients next to several basis elements of the form $\lambda_a^i\,\lambda_b^j\,\lambda_c^k\,\lambda_d^l\,\lambda_e^m$ to get a system of equations. We only consider coefficients next to elements $\lambda_a^i\,\lambda_b^j\,\lambda_c^k\,\lambda_d^l\,\lambda_e^m$ with a,b,c,d,e all distinct to ensure that none of them appears in the expansions of the elements (B.6) corresponding to a product of a 3-cycle and disjoint transposition. From the system

$$\begin{aligned}
\lambda_4^i\,\lambda_1^j\,\lambda_2^k\,\lambda_8^l\,\lambda_5^m:&\quad 2\gamma_1-\gamma_5-\gamma_6+\gamma_7+\gamma_8+\gamma_9-2\gamma_{10}-\gamma_{12}=0,\\
\lambda_4^i\,\lambda_1^j\,\lambda_3^k\,\lambda_5^l\,\lambda_2^m:&\quad \gamma_1-\gamma_2-\gamma_4+\gamma_7-\gamma_{10}+\gamma_{11}-\gamma_{12}=0,\\
\lambda_4^i\,\lambda_1^j\,\lambda_3^k\,\lambda_8^l\,\lambda_7^m:&\quad -2\gamma_1-\gamma_5-\gamma_6+\gamma_7+\gamma_8+\gamma_9-2\gamma_{10}+\gamma_{12}=0,\\
\lambda_4^i\,\lambda_1^j\,\lambda_5^k\,\lambda_8^l\,\lambda_2^m:&\quad -\gamma_1+\gamma_2+\gamma_3+\gamma_4-\gamma_5-\gamma_6-2\gamma_{11}+2\gamma_{12}=0,\\
\lambda_4^i\,\lambda_1^j\,\lambda_6^k\,\lambda_3^l\,\lambda_8^m:&\quad \gamma_1+\gamma_2-\gamma_3+\gamma_4-\gamma_5+\gamma_6-2\gamma_7+2\gamma_8=0,\\
\lambda_4^i\,\lambda_1^j\,\lambda_3^k\,\lambda_7^l\,\lambda_8^m:&\quad \gamma_1+\gamma_2-\gamma_4+2\gamma_6+\gamma_7-2\gamma_9+\gamma_{10}+\gamma_{11}+\gamma_{12}=0,\\
\lambda_4^i\,\lambda_2^j\,\lambda_6^k\,\lambda_8^l\,\lambda_3^m:&\quad -\gamma_1+\gamma_2+\gamma_3-\gamma_4+\gamma_5+\gamma_6-2\gamma_{11}+2\gamma_{12}=0,\\
\lambda_4^i\,\lambda_2^j\,\lambda_8^k\,\lambda_1^l\,\lambda_5^m:&\quad -2\gamma_2+2\gamma_3-2\gamma_4+\gamma_5-\gamma_6+\gamma_7-\gamma_8-\gamma_9+\gamma_{12}=0,\\
\lambda_4^i\,\lambda_3^j\,\lambda_7^k\,\lambda_1^l\,\lambda_8^m:&\quad \gamma_1+\gamma_2-\gamma_3-\gamma_4+\gamma_5-\gamma_6-2\gamma_7+2\gamma_8=0,
\end{aligned}$$

we deduce $\gamma_6=0$. Adding the equations

$$\begin{aligned}
\lambda_4^i\,\lambda_2^j\,\lambda_1^k\,\lambda_5^l\,\lambda_8^m:&\quad \gamma_1+\gamma_2+\gamma_4+\gamma_7+2\gamma_9-\gamma_{10}-\gamma_{11}-\gamma_{12}=0,\\
\lambda_4^i\,\lambda_3^j\,\lambda_8^k\,\lambda_2^l\,\lambda_6^m:&\quad -2\gamma_2+2\gamma_3+2\gamma_4+\gamma_5+\gamma_7-\gamma_8-\gamma_9-\gamma_{12}=0,\\
\lambda_4^i\,\lambda_5^j\,\lambda_1^k\,\lambda_8^l\,\lambda_2^m:&\quad -2\gamma_2+\gamma_3-2\gamma_4+2\gamma_7-\gamma_8-\gamma_{10}+\gamma_{11}=0
\end{aligned}$$

yields $\gamma_9=0$. Moreover, from

$$\lambda_4^i\,\lambda_3^j\,\lambda_8^k\,\lambda_6^l\,\lambda_1^m:\quad \gamma_1-\gamma_2-\gamma_4+2\gamma_5-\gamma_7+\gamma_{10}+\gamma_{11}-\gamma_{12}=0,$$

we obtain $\gamma_7=\gamma_{11}=0$. Finally,

$$\lambda_4^i\,\lambda_2^j\,\lambda_8^k\,\lambda_6^l\,\lambda_3^m:\quad -\gamma_1+\gamma_2-\gamma_4-2\gamma_5+\gamma_{10}-\gamma_{12}=0$$

$$\lambda_4^i\,\lambda_5^j\,\lambda_2^k\,\lambda_3^l\,\lambda_1^m: \quad -\gamma_3+\gamma_8+\gamma_{10}-\gamma_{11}=0$$

yields $\gamma_1=\gamma_2=\gamma_3=\gamma_4=\gamma_5=\gamma_8=\gamma_{10}=\gamma_{12}=0$.

(e) What remains is a linear dependence involving terms of degree at most three, which contradicts Proposition B.2. ■

B.4.1. *Expansions of the remaining* 5*-cycles.* To complete the proof of Proposition B.4 we list the expansions of the 5-cycles not contained in the basis. These were produced with the help of noncommutative Gröbner bases, but can be readily verified by direct matrix calculation.

$$\begin{aligned}
&\mathrm{Swap}_{15}\mathrm{Swap}_{14}\mathrm{Swap}_{13}\mathrm{Swap}_{12} = \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{15}\mathrm{Swap}_{23}\mathrm{Swap}_{24}\\
&+\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{23}\mathrm{Swap}_{25}+\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{34}\mathrm{Swap}_{35}\\
&+\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{24}\mathrm{Swap}_{35}+\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{24}\mathrm{Swap}_{25}\\
&+\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{24}+\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{35}\\
&-\tfrac{1}{2}\,\mathrm{Swap}_{23}\mathrm{Swap}_{34}\mathrm{Swap}_{35}-\tfrac{1}{2}\,\mathrm{Swap}_{23}\mathrm{Swap}_{25}\mathrm{Swap}_{34}-\tfrac{1}{2}\,\mathrm{Swap}_{23}\mathrm{Swap}_{24}\mathrm{Swap}_{45}\\
&-\mathrm{Swap}_{23}\mathrm{Swap}_{24}\mathrm{Swap}_{35}-\tfrac{1}{2}\,\mathrm{Swap}_{23}\mathrm{Swap}_{24}\mathrm{Swap}_{25}+\tfrac{1}{2}\,\mathrm{Swap}_{15}\mathrm{Swap}_{23}\mathrm{Swap}_{34}\\
&+\tfrac{1}{2}\,\mathrm{Swap}_{14}\mathrm{Swap}_{23}\mathrm{Swap}_{45}-\tfrac{1}{2}\,\mathrm{Swap}_{14}\mathrm{Swap}_{23}\mathrm{Swap}_{35}-\tfrac{1}{2}\,\mathrm{Swap}_{14}\mathrm{Swap}_{23}\mathrm{Swap}_{25}\\
&-\mathrm{Swap}_{13}\mathrm{Swap}_{34}\mathrm{Swap}_{35}-\tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{25}\mathrm{Swap}_{34}-\tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{24}\mathrm{Swap}_{45}\\
&-\tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{24}\mathrm{Swap}_{35}-\tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{24}\mathrm{Swap}_{25}-\tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{34}\\
&-\tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{24}-\tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{45}-\tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{35}\\
&-\tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{25}-\tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{15}+\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{34}\mathrm{Swap}_{45}\\
&+\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{25}\mathrm{Swap}_{34}-\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{24}\mathrm{Swap}_{35}-\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{24}\mathrm{Swap}_{25}\\
&+\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{23}\mathrm{Swap}_{45}-\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{23}\mathrm{Swap}_{25}-\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{23}\mathrm{Swap}_{24}\\
&-\mathrm{Swap}_{12}\mathrm{Swap}_{15}\mathrm{Swap}_{24}-\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{15}\mathrm{Swap}_{23}-\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{45}\\
&-\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{35}-\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{25}-\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{23}\\
&-\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{15}-\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{35}-\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{34}\\
&-\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{25}-\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{24}-\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{15}\\
&-\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14}+\tfrac{1}{2}\,\mathrm{Swap}_{34}\mathrm{Swap}_{45}+\mathrm{Swap}_{34}\mathrm{Swap}_{35}+\tfrac{1}{2}\,\mathrm{Swap}_{25}\mathrm{Swap}_{34}\\
&+\mathrm{Swap}_{24}\mathrm{Swap}_{45}+\mathrm{Swap}_{24}\mathrm{Swap}_{35}+\mathrm{Swap}_{24}\mathrm{Swap}_{25}+\mathrm{Swap}_{23}\mathrm{Swap}_{35}\\
&+\tfrac{1}{2}\,\mathrm{Swap}_{23}\mathrm{Swap}_{34}+\mathrm{Swap}_{23}\mathrm{Swap}_{25}+\mathrm{Swap}_{23}\mathrm{Swap}_{24}+\tfrac{1}{2}\,\mathrm{Swap}_{15}\mathrm{Swap}_{24}\\
&+\tfrac{1}{2}\,\mathrm{Swap}_{14}\mathrm{Swap}_{45}+\mathrm{Swap}_{14}\mathrm{Swap}_{35}+\mathrm{Swap}_{14}\mathrm{Swap}_{25}+\tfrac{1}{2}\,\mathrm{Swap}_{14}\mathrm{Swap}_{23}\\
&+\mathrm{Swap}_{14}\mathrm{Swap}_{15}+\tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{45}+\mathrm{Swap}_{13}\mathrm{Swap}_{35}+\mathrm{Swap}_{13}\mathrm{Swap}_{34}\\
&+\mathrm{Swap}_{13}\mathrm{Swap}_{25}+\mathrm{Swap}_{13}\mathrm{Swap}_{24}+\mathrm{Swap}_{13}\mathrm{Swap}_{15}+\mathrm{Swap}_{13}\mathrm{Swap}_{14}\\
&+\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{35}+\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{25}+\mathrm{Swap}_{12}\mathrm{Swap}_{24}+\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{23}\\
&+\mathrm{Swap}_{12}\mathrm{Swap}_{15}+\mathrm{Swap}_{12}\mathrm{Swap}_{14}+\mathrm{Swap}_{12}\mathrm{Swap}_{13}-\mathrm{Swap}_{45}-\tfrac{3}{2}\,\mathrm{Swap}_{35}-\mathrm{Swap}_{34}\\
&-\tfrac{3}{2}\,\mathrm{Swap}_{25}-\tfrac{3}{2}\,\mathrm{Swap}_{24}-\mathrm{Swap}_{23}-\mathrm{Swap}_{15}-\tfrac{3}{2}\,\mathrm{Swap}_{14}-\tfrac{3}{2}\,\mathrm{Swap}_{13}-\mathrm{Swap}_{12}+2
\end{aligned}$$

$$\begin{aligned}
&\mathrm{Swap}_{14}\mathrm{Swap}_{15}\mathrm{Swap}_{13}\mathrm{Swap}_{12} = \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{15}\mathrm{Swap}_{23}\mathrm{Swap}_{24}\\
&+\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{23}\mathrm{Swap}_{25}-\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{34}\mathrm{Swap}_{35}
\end{aligned}$$

$$
\begin{aligned}
&- \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{24}\mathrm{Swap}_{35} - \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{24}\mathrm{Swap}_{25}\\
&+ \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{34} + \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{24}\\
&+ \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{45} + \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{35}\\
&+ \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{25} + \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{15}\\
&+ \tfrac{1}{2}\,\mathrm{Swap}_{23}\mathrm{Swap}_{34}\mathrm{Swap}_{35} - \tfrac{1}{2}\,\mathrm{Swap}_{23}\mathrm{Swap}_{25}\mathrm{Swap}_{34} - \tfrac{1}{2}\,\mathrm{Swap}_{23}\mathrm{Swap}_{24}\mathrm{Swap}_{45}\\
&- \tfrac{1}{2}\,\mathrm{Swap}_{23}\mathrm{Swap}_{24}\mathrm{Swap}_{25} - \tfrac{1}{2}\,\mathrm{Swap}_{15}\mathrm{Swap}_{23}\mathrm{Swap}_{34} - \mathrm{Swap}_{15}\mathrm{Swap}_{23}\mathrm{Swap}_{24}\\
&- \tfrac{1}{2}\,\mathrm{Swap}_{14}\mathrm{Swap}_{23}\mathrm{Swap}_{45} + \tfrac{1}{2}\,\mathrm{Swap}_{14}\mathrm{Swap}_{23}\mathrm{Swap}_{35} - \tfrac{1}{2}\,\mathrm{Swap}_{14}\mathrm{Swap}_{23}\mathrm{Swap}_{25}\\
&+ \mathrm{Swap}_{13}\mathrm{Swap}_{34}\mathrm{Swap}_{35} + \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{25}\mathrm{Swap}_{34} + \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{24}\mathrm{Swap}_{45}\\
&+ \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{24}\mathrm{Swap}_{35} + \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{24}\mathrm{Swap}_{25} - \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{34}\\
&- \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{24} - \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{45} - \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{35}\\
&- \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{25} - \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{15} - \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{34}\mathrm{Swap}_{45}\\
&- \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{25}\mathrm{Swap}_{34} + \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{24}\mathrm{Swap}_{35} + \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{24}\mathrm{Swap}_{25}\\
&+ \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{23}\mathrm{Swap}_{45} - \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{23}\mathrm{Swap}_{25} - \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{23}\mathrm{Swap}_{24}\\
&- \mathrm{Swap}_{12}\mathrm{Swap}_{15}\mathrm{Swap}_{34} - \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{15}\mathrm{Swap}_{23} - \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{45}\\
&- \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{35} - \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{25} - \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{23}\\
&- \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{15} - \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{34} - \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{25}\\
&- \tfrac{3}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{15} - \tfrac{3}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14} + \tfrac{1}{2}\,\mathrm{Swap}_{34}\mathrm{Swap}_{45}\\
&+ \tfrac{1}{2}\,\mathrm{Swap}_{25}\mathrm{Swap}_{34} + \tfrac{1}{2}\,\mathrm{Swap}_{23}\mathrm{Swap}_{34} + \mathrm{Swap}_{23}\mathrm{Swap}_{25} + \mathrm{Swap}_{23}\mathrm{Swap}_{24}\\
&+ \mathrm{Swap}_{15}\mathrm{Swap}_{34} + \tfrac{1}{2}\,\mathrm{Swap}_{15}\mathrm{Swap}_{24} + \mathrm{Swap}_{15}\mathrm{Swap}_{23} + \tfrac{1}{2}\,\mathrm{Swap}_{14}\mathrm{Swap}_{45}\\
&+ \tfrac{1}{2}\,\mathrm{Swap}_{14}\mathrm{Swap}_{23} - \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{45} + \mathrm{Swap}_{13}\mathrm{Swap}_{15} + \mathrm{Swap}_{13}\mathrm{Swap}_{14}\\
&+ \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{35} + \mathrm{Swap}_{12}\mathrm{Swap}_{34} + \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{25} + \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{23}\\
&+ \mathrm{Swap}_{12}\mathrm{Swap}_{15} + \mathrm{Swap}_{12}\mathrm{Swap}_{14} + \mathrm{Swap}_{12}\mathrm{Swap}_{13} - \tfrac{1}{2}\,\mathrm{Swap}_{35} - \mathrm{Swap}_{34} - \tfrac{1}{2}\,\mathrm{Swap}_{25}\\
&- \tfrac{1}{2}\,\mathrm{Swap}_{24} - \mathrm{Swap}_{23} - \mathrm{Swap}_{15} - \tfrac{1}{2}\,\mathrm{Swap}_{14} - \tfrac{1}{2}\,\mathrm{Swap}_{13} - \mathrm{Swap}_{12} + 1
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{Swap}_{15}\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{12} = -\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{15}\mathrm{Swap}_{23}\mathrm{Swap}_{24}\\
&+ \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{23}\mathrm{Swap}_{25} + \mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{15}\mathrm{Swap}_{23}\\
&+ \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{34}\mathrm{Swap}_{35} - \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{24}\mathrm{Swap}_{35}\\
&+ \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{24}\mathrm{Swap}_{25} + \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{34}\\
&+ \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{24} + \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{45}\\
&- \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{35} + \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{15}\\
&- \tfrac{1}{2}\,\mathrm{Swap}_{23}\mathrm{Swap}_{34}\mathrm{Swap}_{35} - \tfrac{1}{2}\,\mathrm{Swap}_{23}\mathrm{Swap}_{25}\mathrm{Swap}_{34} - \tfrac{1}{2}\,\mathrm{Swap}_{23}\mathrm{Swap}_{24}\mathrm{Swap}_{45}\\
&+ \mathrm{Swap}_{23}\mathrm{Swap}_{24}\mathrm{Swap}_{35} - \tfrac{1}{2}\,\mathrm{Swap}_{23}\mathrm{Swap}_{24}\mathrm{Swap}_{25} - \tfrac{1}{2}\,\mathrm{Swap}_{15}\mathrm{Swap}_{23}\mathrm{Swap}_{34}\\
&- \tfrac{1}{2}\,\mathrm{Swap}_{14}\mathrm{Swap}_{23}\mathrm{Swap}_{45} + \tfrac{1}{2}\,\mathrm{Swap}_{14}\mathrm{Swap}_{23}\mathrm{Swap}_{35} - \tfrac{1}{2}\,\mathrm{Swap}_{14}\mathrm{Swap}_{23}\mathrm{Swap}_{25}\\
&- \mathrm{Swap}_{14}\mathrm{Swap}_{15}\mathrm{Swap}_{23} + \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{25}\mathrm{Swap}_{34} + \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{24}\mathrm{Swap}_{45}\\
&+ \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{24}\mathrm{Swap}_{35} - \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{24}\mathrm{Swap}_{25} - \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{34}\\
&- \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{24} - \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{45} + \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{35}
\end{aligned}
$$

$$+ \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{25} - \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{15} - \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{34}\mathrm{Swap}_{45}$$
$$+ \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{25}\mathrm{Swap}_{34} + \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{24}\mathrm{Swap}_{35} - \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{24}\mathrm{Swap}_{25}$$
$$- \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{23}\mathrm{Swap}_{45} - \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{23}\mathrm{Swap}_{25} + \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{23}\mathrm{Swap}_{24}$$
$$- \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{15}\mathrm{Swap}_{23} - \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{45} - \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{35}$$
$$- \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{25} - \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{23} - \tfrac{3}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{15}$$
$$- \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{45} - \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{34} - \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{25}$$
$$- \tfrac{3}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{15} - \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14} + \tfrac{1}{2}\,\mathrm{Swap}_{34}\mathrm{Swap}_{45}$$
$$- \tfrac{1}{2}\,\mathrm{Swap}_{25}\mathrm{Swap}_{34} + \mathrm{Swap}_{24}\mathrm{Swap}_{25} + \mathrm{Swap}_{23}\mathrm{Swap}_{45} + \tfrac{1}{2}\,\mathrm{Swap}_{23}\mathrm{Swap}_{34}$$
$$+ \mathrm{Swap}_{23}\mathrm{Swap}_{25} + \tfrac{1}{2}\,\mathrm{Swap}_{15}\mathrm{Swap}_{24} + \mathrm{Swap}_{15}\mathrm{Swap}_{23} + \tfrac{1}{2}\,\mathrm{Swap}_{14}\mathrm{Swap}_{45}$$
$$+ \tfrac{1}{2}\,\mathrm{Swap}_{14}\mathrm{Swap}_{23} + \mathrm{Swap}_{14}\mathrm{Swap}_{15} + \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{45} + \mathrm{Swap}_{13}\mathrm{Swap}_{15}$$
$$+ \mathrm{Swap}_{12}\mathrm{Swap}_{45} + \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{35} + \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{25} + \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{23}$$
$$+ \mathrm{Swap}_{12}\mathrm{Swap}_{15} + \mathrm{Swap}_{12}\mathrm{Swap}_{14} + \mathrm{Swap}_{12}\mathrm{Swap}_{13} - \mathrm{Swap}_{45} - \tfrac{1}{2}\,\mathrm{Swap}_{35}$$
$$- \tfrac{1}{2}\,\mathrm{Swap}_{25} - \tfrac{1}{2}\,\mathrm{Swap}_{24} - \mathrm{Swap}_{23} - \mathrm{Swap}_{15} - \tfrac{1}{2}\,\mathrm{Swap}_{14} - \tfrac{1}{2}\,\mathrm{Swap}_{13} - \mathrm{Swap}_{12} + 1$$

$$\mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{14}\mathrm{Swap}_{12} = -\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{15}\mathrm{Swap}_{23}\mathrm{Swap}_{24}$$
$$- \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{23}\mathrm{Swap}_{25} - \mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{15}\mathrm{Swap}_{23}$$
$$- \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{34}\mathrm{Swap}_{35} + \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{24}\mathrm{Swap}_{25}$$
$$- \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{34} - \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{24}$$
$$+ \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{35} + \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{25}$$
$$+ \tfrac{1}{2}\,\mathrm{Swap}_{23}\mathrm{Swap}_{34}\mathrm{Swap}_{35} + \tfrac{1}{2}\,\mathrm{Swap}_{23}\mathrm{Swap}_{25}\mathrm{Swap}_{34} + \tfrac{1}{2}\,\mathrm{Swap}_{23}\mathrm{Swap}_{24}\mathrm{Swap}_{45}$$
$$- \tfrac{1}{2}\,\mathrm{Swap}_{23}\mathrm{Swap}_{24}\mathrm{Swap}_{25} + \tfrac{1}{2}\,\mathrm{Swap}_{15}\mathrm{Swap}_{23}\mathrm{Swap}_{34} - \tfrac{1}{2}\,\mathrm{Swap}_{14}\mathrm{Swap}_{23}\mathrm{Swap}_{45}$$
$$- \tfrac{1}{2}\,\mathrm{Swap}_{14}\mathrm{Swap}_{23}\mathrm{Swap}_{35} - \tfrac{1}{2}\,\mathrm{Swap}_{14}\mathrm{Swap}_{23}\mathrm{Swap}_{25} - \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{25}\mathrm{Swap}_{34}$$
$$+ \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{24}\mathrm{Swap}_{45} + \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{24}\mathrm{Swap}_{35} + \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{24}\mathrm{Swap}_{25}$$
$$+ \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{34} + \tfrac{3}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{24} + \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{45}$$
$$- \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{35} - \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{25} + \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{15}$$
$$+ \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{34}\mathrm{Swap}_{45} - \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{25}\mathrm{Swap}_{34} + \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{24}\mathrm{Swap}_{35}$$
$$- \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{24}\mathrm{Swap}_{25} + \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{23}\mathrm{Swap}_{45} + \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{23}\mathrm{Swap}_{25}$$
$$+ \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{23}\mathrm{Swap}_{24} + \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{15}\mathrm{Swap}_{23} - \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{45}$$
$$+ \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{35} - \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{25} + \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{23}$$
$$+ \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{15} + \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{34} - \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{25}$$
$$+ \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{15} - \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14} - \tfrac{1}{2}\,\mathrm{Swap}_{34}\mathrm{Swap}_{45}$$
$$+ \tfrac{1}{2}\,\mathrm{Swap}_{25}\mathrm{Swap}_{34} - \mathrm{Swap}_{24}\mathrm{Swap}_{35} - \tfrac{1}{2}\,\mathrm{Swap}_{23}\mathrm{Swap}_{34} - \tfrac{1}{2}\,\mathrm{Swap}_{15}\mathrm{Swap}_{24}$$
$$+ \tfrac{1}{2}\,\mathrm{Swap}_{14}\mathrm{Swap}_{45} + \mathrm{Swap}_{14}\mathrm{Swap}_{25} + \tfrac{1}{2}\,\mathrm{Swap}_{14}\mathrm{Swap}_{23} - \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{45}$$
$$- \mathrm{Swap}_{13}\mathrm{Swap}_{24} - \mathrm{Swap}_{13}\mathrm{Swap}_{15} - \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{35} + \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{25}$$
$$- \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{23} + \tfrac{1}{2}\,\mathrm{Swap}_{35} - \tfrac{1}{2}\,\mathrm{Swap}_{25} + \tfrac{1}{2}\,\mathrm{Swap}_{24} - \tfrac{1}{2}\,\mathrm{Swap}_{14} + \tfrac{1}{2}\,\mathrm{Swap}_{13}$$

$$\begin{aligned}
&\mathrm{Swap}_{14}\mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{12} = \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{15}\mathrm{Swap}_{23}\mathrm{Swap}_{24}\\
&-\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{23}\mathrm{Swap}_{25} + \mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{15}\mathrm{Swap}_{23}\\
&+\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{34}\mathrm{Swap}_{35} - \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{24}\mathrm{Swap}_{25}\\
&-\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{24} - \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{45}\\
&-\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{35} - \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{25}\\
&-\tfrac{1}{2}\,\mathrm{Swap}_{23}\mathrm{Swap}_{34}\mathrm{Swap}_{35} + \tfrac{1}{2}\,\mathrm{Swap}_{23}\mathrm{Swap}_{25}\mathrm{Swap}_{34} + \tfrac{1}{2}\,\mathrm{Swap}_{23}\mathrm{Swap}_{24}\mathrm{Swap}_{45}\\
&+\tfrac{1}{2}\,\mathrm{Swap}_{23}\mathrm{Swap}_{24}\mathrm{Swap}_{25} - \tfrac{1}{2}\,\mathrm{Swap}_{15}\mathrm{Swap}_{23}\mathrm{Swap}_{34} + \tfrac{1}{2}\,\mathrm{Swap}_{14}\mathrm{Swap}_{23}\mathrm{Swap}_{45}\\
&+\tfrac{1}{2}\,\mathrm{Swap}_{14}\mathrm{Swap}_{23}\mathrm{Swap}_{35} + \tfrac{3}{2}\,\mathrm{Swap}_{14}\mathrm{Swap}_{23}\mathrm{Swap}_{25} + \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{25}\mathrm{Swap}_{34}\\
&-\tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{24}\mathrm{Swap}_{45} - \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{24}\mathrm{Swap}_{35} - \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{24}\mathrm{Swap}_{25}\\
&+\tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{34} - \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{24} + \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{45}\\
&+\tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{35} + \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{25} - \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{15}\\
&+\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{34}\mathrm{Swap}_{45} + \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{25}\mathrm{Swap}_{34} + \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{24}\mathrm{Swap}_{35}\\
&+\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{24}\mathrm{Swap}_{25} - \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{23}\mathrm{Swap}_{45} + \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{23}\mathrm{Swap}_{25}\\
&-\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{23}\mathrm{Swap}_{24} - \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{15}\mathrm{Swap}_{23} + \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{45}\\
&+\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{35} + \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{25} - \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{23}\\
&-\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{15} - \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{34} + \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{25}\\
&+\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{15} + \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14} - \tfrac{1}{2}\,\mathrm{Swap}_{34}\mathrm{Swap}_{45}\\
&-\tfrac{1}{2}\,\mathrm{Swap}_{25}\mathrm{Swap}_{34} + \tfrac{1}{2}\,\mathrm{Swap}_{23}\mathrm{Swap}_{34} - \mathrm{Swap}_{23}\mathrm{Swap}_{25} + \tfrac{1}{2}\,\mathrm{Swap}_{15}\mathrm{Swap}_{24}\\
&-\tfrac{1}{2}\,\mathrm{Swap}_{14}\mathrm{Swap}_{45} - \mathrm{Swap}_{14}\mathrm{Swap}_{35} - \mathrm{Swap}_{14}\mathrm{Swap}_{25} - \tfrac{1}{2}\,\mathrm{Swap}_{14}\mathrm{Swap}_{23}\\
&+\tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{45} + \mathrm{Swap}_{13}\mathrm{Swap}_{24} - \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{35} - \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{25}\\
&+\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{23} + \tfrac{1}{2}\,\mathrm{Swap}_{35} + \tfrac{1}{2}\,\mathrm{Swap}_{25} - \tfrac{1}{2}\,\mathrm{Swap}_{24} + \tfrac{1}{2}\,\mathrm{Swap}_{14} - \tfrac{1}{2}\,\mathrm{Swap}_{13}
\end{aligned}$$

$$\begin{aligned}
&\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{15}\mathrm{Swap}_{12} = -\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{15}\mathrm{Swap}_{23}\mathrm{Swap}_{24}\\
&-\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{23}\mathrm{Swap}_{25} - \mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{15}\mathrm{Swap}_{23}\\
&-\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{34}\mathrm{Swap}_{35} + \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{24}\mathrm{Swap}_{35}\\
&-\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{24}\mathrm{Swap}_{25} - \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{34}\\
&-\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{24} - \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{45}\\
&-\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{35} - \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{25}\\
&-2\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{15} + \tfrac{1}{2}\,\mathrm{Swap}_{23}\mathrm{Swap}_{34}\mathrm{Swap}_{35}\\
&+\tfrac{1}{2}\,\mathrm{Swap}_{23}\mathrm{Swap}_{25}\mathrm{Swap}_{34} + \tfrac{1}{2}\,\mathrm{Swap}_{23}\mathrm{Swap}_{24}\mathrm{Swap}_{45} + \tfrac{3}{2}\,\mathrm{Swap}_{23}\mathrm{Swap}_{24}\mathrm{Swap}_{25}\\
&+\tfrac{1}{2}\,\mathrm{Swap}_{15}\mathrm{Swap}_{23}\mathrm{Swap}_{34} + \mathrm{Swap}_{15}\mathrm{Swap}_{23}\mathrm{Swap}_{24} + \tfrac{1}{2}\,\mathrm{Swap}_{14}\mathrm{Swap}_{23}\mathrm{Swap}_{45}\\
&-\tfrac{1}{2}\,\mathrm{Swap}_{14}\mathrm{Swap}_{23}\mathrm{Swap}_{35} + \tfrac{1}{2}\,\mathrm{Swap}_{14}\mathrm{Swap}_{23}\mathrm{Swap}_{25} + \mathrm{Swap}_{14}\mathrm{Swap}_{15}\mathrm{Swap}_{23}\\
&-\tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{25}\mathrm{Swap}_{34} - \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{24}\mathrm{Swap}_{45} - \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{24}\mathrm{Swap}_{35}\\
&+\tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{24}\mathrm{Swap}_{25} + \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{34} + \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{24}\\
&+\tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{45} + \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{35} + \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{25}\\
&+\tfrac{3}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{15} + \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{34}\mathrm{Swap}_{45} + \mathrm{Swap}_{12}\mathrm{Swap}_{34}\mathrm{Swap}_{35}
\end{aligned}$$

$$+\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{25}\mathrm{Swap}_{34} - \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{24}\mathrm{Swap}_{35} + \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{24}\mathrm{Swap}_{25}$$
$$+\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{23}\mathrm{Swap}_{45} + \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{23}\mathrm{Swap}_{25} + \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{23}\mathrm{Swap}_{24}$$
$$+\mathrm{Swap}_{12}\mathrm{Swap}_{15}\mathrm{Swap}_{34} + \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{15}\mathrm{Swap}_{23} + \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{45}$$
$$+\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{35} + \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{25} + \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{23}$$
$$+\tfrac{3}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{15} + \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{45} + \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{34}$$
$$+\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{25} + \tfrac{3}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{15} + \tfrac{3}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14}$$
$$-\tfrac{1}{2}\,\mathrm{Swap}_{34}\mathrm{Swap}_{45} - \mathrm{Swap}_{34}\mathrm{Swap}_{35} - \tfrac{1}{2}\,\mathrm{Swap}_{25}\mathrm{Swap}_{34} - \mathrm{Swap}_{24}\mathrm{Swap}_{25}$$
$$-\mathrm{Swap}_{23}\mathrm{Swap}_{45} - \tfrac{1}{2}\,\mathrm{Swap}_{23}\mathrm{Swap}_{34} - \mathrm{Swap}_{23}\mathrm{Swap}_{25} - \mathrm{Swap}_{23}\mathrm{Swap}_{24}$$
$$-\mathrm{Swap}_{15}\mathrm{Swap}_{34} - \tfrac{1}{2}\,\mathrm{Swap}_{15}\mathrm{Swap}_{24} - \mathrm{Swap}_{15}\mathrm{Swap}_{23} - \tfrac{1}{2}\,\mathrm{Swap}_{14}\mathrm{Swap}_{45}$$
$$-\tfrac{1}{2}\,\mathrm{Swap}_{14}\mathrm{Swap}_{23} - \mathrm{Swap}_{14}\mathrm{Swap}_{15} - \tfrac{1}{2}\,\mathrm{Swap}_{13}\mathrm{Swap}_{45} - \mathrm{Swap}_{13}\mathrm{Swap}_{15}$$
$$-\mathrm{Swap}_{13}\mathrm{Swap}_{14} - \mathrm{Swap}_{12}\mathrm{Swap}_{45} - \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{35} - \mathrm{Swap}_{12}\mathrm{Swap}_{34}$$
$$-\tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{25} - \tfrac{1}{2}\,\mathrm{Swap}_{12}\mathrm{Swap}_{23} - \mathrm{Swap}_{12}\mathrm{Swap}_{15} - \mathrm{Swap}_{12}\mathrm{Swap}_{14}$$
$$-\mathrm{Swap}_{12}\mathrm{Swap}_{13} + \mathrm{Swap}_{45} + \tfrac{1}{2}\,\mathrm{Swap}_{35} + \mathrm{Swap}_{34} + \tfrac{1}{2}\,\mathrm{Swap}_{25} + \tfrac{1}{2}\,\mathrm{Swap}_{24}$$
$$+\mathrm{Swap}_{23} + \mathrm{Swap}_{15} + \tfrac{1}{2}\,\mathrm{Swap}_{14} + \tfrac{1}{2}\,\mathrm{Swap}_{13} + \mathrm{Swap}_{12} - 1$$

$$\mathrm{Swap}_{15}\mathrm{Swap}_{14}\mathrm{Swap}_{12}\mathrm{Swap}_{13} = -\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{24}\mathrm{Swap}_{25}$$
$$-\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{24} - \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{45}$$
$$-\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{25} - \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{15}$$
$$+\mathrm{Swap}_{23}\mathrm{Swap}_{24}\mathrm{Swap}_{45} + \mathrm{Swap}_{23}\mathrm{Swap}_{24}\mathrm{Swap}_{25} + \mathrm{Swap}_{15}\mathrm{Swap}_{23}\mathrm{Swap}_{24}$$
$$+\mathrm{Swap}_{14}\mathrm{Swap}_{23}\mathrm{Swap}_{45} + \mathrm{Swap}_{14}\mathrm{Swap}_{23}\mathrm{Swap}_{25} + \mathrm{Swap}_{14}\mathrm{Swap}_{15}\mathrm{Swap}_{23}$$
$$+\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{45} + \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{25} + \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{24}$$
$$+\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{15} + \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14} - \mathrm{Swap}_{23}\mathrm{Swap}_{45}$$
$$-\mathrm{Swap}_{23}\mathrm{Swap}_{25} - \mathrm{Swap}_{23}\mathrm{Swap}_{24} - \mathrm{Swap}_{15}\mathrm{Swap}_{23} - \mathrm{Swap}_{14}\mathrm{Swap}_{23}$$
$$-\mathrm{Swap}_{12}\mathrm{Swap}_{13} + \mathrm{Swap}_{23}$$

$$\mathrm{Swap}_{15}\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{13} = -\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{24}\mathrm{Swap}_{35}$$
$$+\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{34} + \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{24}$$
$$+2\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{45} + \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{35}$$
$$+\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{25} + 2\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{15}$$
$$-\mathrm{Swap}_{23}\mathrm{Swap}_{24}\mathrm{Swap}_{45} - \mathrm{Swap}_{23}\mathrm{Swap}_{24}\mathrm{Swap}_{25} - \mathrm{Swap}_{15}\mathrm{Swap}_{23}\mathrm{Swap}_{24}$$
$$-\mathrm{Swap}_{14}\mathrm{Swap}_{23}\mathrm{Swap}_{45} - \mathrm{Swap}_{14}\mathrm{Swap}_{23}\mathrm{Swap}_{25} - \mathrm{Swap}_{14}\mathrm{Swap}_{15}\mathrm{Swap}_{23}$$
$$+\mathrm{Swap}_{13}\mathrm{Swap}_{25}\mathrm{Swap}_{34} + \mathrm{Swap}_{13}\mathrm{Swap}_{24}\mathrm{Swap}_{45} + \mathrm{Swap}_{13}\mathrm{Swap}_{24}\mathrm{Swap}_{35}$$
$$+\mathrm{Swap}_{13}\mathrm{Swap}_{24}\mathrm{Swap}_{25} - \mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{34} - \mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{45}$$
$$-\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{35} - \mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{15} - \mathrm{Swap}_{12}\mathrm{Swap}_{34}\mathrm{Swap}_{45}$$
$$-\mathrm{Swap}_{12}\mathrm{Swap}_{34}\mathrm{Swap}_{35} - \mathrm{Swap}_{12}\mathrm{Swap}_{15}\mathrm{Swap}_{34} - \mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{45}$$
$$-\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{35} - \mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{15} - \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{45}$$
$$-2\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{15} - 2\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14} + \mathrm{Swap}_{34}\mathrm{Swap}_{45}$$

$$
\begin{aligned}
&+ \mathrm{Swap}_{34}\mathrm{Swap}_{35} + \mathrm{Swap}_{23}\mathrm{Swap}_{45} + \mathrm{Swap}_{23}\mathrm{Swap}_{25} + \mathrm{Swap}_{23}\mathrm{Swap}_{24}\\
&+ \mathrm{Swap}_{15}\mathrm{Swap}_{34} + \mathrm{Swap}_{15}\mathrm{Swap}_{23} + \mathrm{Swap}_{14}\mathrm{Swap}_{45} + \mathrm{Swap}_{14}\mathrm{Swap}_{35}\\
&+ \mathrm{Swap}_{14}\mathrm{Swap}_{23} + \mathrm{Swap}_{14}\mathrm{Swap}_{15} - \mathrm{Swap}_{13}\mathrm{Swap}_{25} - \mathrm{Swap}_{13}\mathrm{Swap}_{24}\\
&+ \mathrm{Swap}_{13}\mathrm{Swap}_{15} + \mathrm{Swap}_{13}\mathrm{Swap}_{14} + \mathrm{Swap}_{12}\mathrm{Swap}_{45} + \mathrm{Swap}_{12}\mathrm{Swap}_{35}\\
&+ \mathrm{Swap}_{12}\mathrm{Swap}_{34} + \mathrm{Swap}_{12}\mathrm{Swap}_{15} + \mathrm{Swap}_{12}\mathrm{Swap}_{14} + \mathrm{Swap}_{12}\mathrm{Swap}_{13}\\
&- \mathrm{Swap}_{45} - \mathrm{Swap}_{35} - \mathrm{Swap}_{34} - \mathrm{Swap}_{23} - \mathrm{Swap}_{15} - \mathrm{Swap}_{14} - \mathrm{Swap}_{12} + 1
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{Swap}_{12}\mathrm{Swap}_{15}\mathrm{Swap}_{14}\mathrm{Swap}_{13} = -\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{34}\mathrm{Swap}_{35}\\
&- \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{34} - \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{45}\\
&- \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{35} - \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{15}\\
&+ \mathrm{Swap}_{12}\mathrm{Swap}_{34}\mathrm{Swap}_{45} + \mathrm{Swap}_{12}\mathrm{Swap}_{34}\mathrm{Swap}_{35} + \mathrm{Swap}_{12}\mathrm{Swap}_{15}\mathrm{Swap}_{34}\\
&+ \mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{45} + \mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{35} + \mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{15}\\
&+ \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{45} + \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{35} + \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{34}\\
&+ \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{15} + \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14} - \mathrm{Swap}_{12}\mathrm{Swap}_{45}\\
&- \mathrm{Swap}_{12}\mathrm{Swap}_{35} - \mathrm{Swap}_{12}\mathrm{Swap}_{34} - \mathrm{Swap}_{12}\mathrm{Swap}_{15} - \mathrm{Swap}_{12}\mathrm{Swap}_{14}\\
&- \mathrm{Swap}_{12}\mathrm{Swap}_{13} + \mathrm{Swap}_{12}
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{Swap}_{15}\mathrm{Swap}_{13}\mathrm{Swap}_{12}\mathrm{Swap}_{14} = -\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{23}\mathrm{Swap}_{25}\\
&- \mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{15}\mathrm{Swap}_{23} - \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{34}\mathrm{Swap}_{35}\\
&+ \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{24}\mathrm{Swap}_{35} - 2\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{34}\\
&- \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{24} - 2\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{45}\\
&- \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{35} - \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{25}\\
&- 2\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{15} + \mathrm{Swap}_{23}\mathrm{Swap}_{34}\mathrm{Swap}_{35}\\
&+ \mathrm{Swap}_{23}\mathrm{Swap}_{25}\mathrm{Swap}_{34} + \mathrm{Swap}_{23}\mathrm{Swap}_{24}\mathrm{Swap}_{45} + \mathrm{Swap}_{23}\mathrm{Swap}_{24}\mathrm{Swap}_{25}\\
&+ \mathrm{Swap}_{15}\mathrm{Swap}_{23}\mathrm{Swap}_{34} + \mathrm{Swap}_{15}\mathrm{Swap}_{23}\mathrm{Swap}_{24} + \mathrm{Swap}_{14}\mathrm{Swap}_{23}\mathrm{Swap}_{45}\\
&+ \mathrm{Swap}_{14}\mathrm{Swap}_{23}\mathrm{Swap}_{25} + \mathrm{Swap}_{14}\mathrm{Swap}_{15}\mathrm{Swap}_{23} - \mathrm{Swap}_{13}\mathrm{Swap}_{25}\mathrm{Swap}_{34}\\
&- \mathrm{Swap}_{13}\mathrm{Swap}_{24}\mathrm{Swap}_{45} + \mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{34} + \mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{24}\\
&+ \mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{45} + \mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{35} + \mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{15}\\
&+ \mathrm{Swap}_{12}\mathrm{Swap}_{34}\mathrm{Swap}_{45} + \mathrm{Swap}_{12}\mathrm{Swap}_{34}\mathrm{Swap}_{35} + \mathrm{Swap}_{12}\mathrm{Swap}_{15}\mathrm{Swap}_{34}\\
&+ \mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{45} + 2\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{35} + \mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{25}\\
&+ \mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{23} + 2\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{15} + \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{45}\\
&+ \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{34} + 2\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{15} + 2\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14}\\
&- \mathrm{Swap}_{34}\mathrm{Swap}_{45} - \mathrm{Swap}_{34}\mathrm{Swap}_{35} - \mathrm{Swap}_{24}\mathrm{Swap}_{35} - \mathrm{Swap}_{24}\mathrm{Swap}_{25}\\
&- \mathrm{Swap}_{23}\mathrm{Swap}_{45} - \mathrm{Swap}_{23}\mathrm{Swap}_{34} - \mathrm{Swap}_{23}\mathrm{Swap}_{25} - \mathrm{Swap}_{23}\mathrm{Swap}_{24}\\
&- \mathrm{Swap}_{15}\mathrm{Swap}_{34} - \mathrm{Swap}_{15}\mathrm{Swap}_{24} - \mathrm{Swap}_{15}\mathrm{Swap}_{23} - \mathrm{Swap}_{14}\mathrm{Swap}_{45}\\
&- \mathrm{Swap}_{14}\mathrm{Swap}_{35} - \mathrm{Swap}_{14}\mathrm{Swap}_{23} - \mathrm{Swap}_{14}\mathrm{Swap}_{15} + \mathrm{Swap}_{13}\mathrm{Swap}_{25}\\
&- \mathrm{Swap}_{13}\mathrm{Swap}_{15} - \mathrm{Swap}_{13}\mathrm{Swap}_{14} - \mathrm{Swap}_{12}\mathrm{Swap}_{45} - \mathrm{Swap}_{12}\mathrm{Swap}_{35}
\end{aligned}
$$

$$- \mathrm{Swap}_{12}\mathrm{Swap}_{34} - \mathrm{Swap}_{12}\mathrm{Swap}_{15} - 2\,\mathrm{Swap}_{12}\mathrm{Swap}_{14} - \mathrm{Swap}_{12}\mathrm{Swap}_{13}$$
$$+ \mathrm{Swap}_{45} + \mathrm{Swap}_{35} + \mathrm{Swap}_{34} + \mathrm{Swap}_{24} + \mathrm{Swap}_{23} + \mathrm{Swap}_{15} + \mathrm{Swap}_{14} + \mathrm{Swap}_{12} - 1$$

$$\mathrm{Swap}_{13}\mathrm{Swap}_{12}\mathrm{Swap}_{15}\mathrm{Swap}_{14} = \mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{15}\mathrm{Swap}_{23}$$
$$+ \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{34}\mathrm{Swap}_{35} - \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{24}\mathrm{Swap}_{35}$$
$$+ 2\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{34} + \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{24}$$
$$+ \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{45} + 2\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{15}$$
$$- \mathrm{Swap}_{23}\mathrm{Swap}_{34}\mathrm{Swap}_{35} - \mathrm{Swap}_{23}\mathrm{Swap}_{25}\mathrm{Swap}_{34} - \mathrm{Swap}_{23}\mathrm{Swap}_{24}\mathrm{Swap}_{45}$$
$$- \mathrm{Swap}_{23}\mathrm{Swap}_{24}\mathrm{Swap}_{25} - \mathrm{Swap}_{15}\mathrm{Swap}_{23}\mathrm{Swap}_{34} - \mathrm{Swap}_{15}\mathrm{Swap}_{23}\mathrm{Swap}_{24}$$
$$+ \mathrm{Swap}_{14}\mathrm{Swap}_{23}\mathrm{Swap}_{35} - \mathrm{Swap}_{14}\mathrm{Swap}_{15}\mathrm{Swap}_{23} + \mathrm{Swap}_{13}\mathrm{Swap}_{25}\mathrm{Swap}_{34}$$
$$+ \mathrm{Swap}_{13}\mathrm{Swap}_{24}\mathrm{Swap}_{45} - \mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{34} - \mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{24}$$
$$+ \mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{25} - \mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{15} - \mathrm{Swap}_{12}\mathrm{Swap}_{34}\mathrm{Swap}_{45}$$
$$- \mathrm{Swap}_{12}\mathrm{Swap}_{34}\mathrm{Swap}_{35} - \mathrm{Swap}_{12}\mathrm{Swap}_{15}\mathrm{Swap}_{34} - \mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{35}$$
$$- 2\,\mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{15} - \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{45} - \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{34}$$
$$- 2\,\mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{15} - \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14} + \mathrm{Swap}_{34}\mathrm{Swap}_{45}$$
$$+ \mathrm{Swap}_{34}\mathrm{Swap}_{35} + \mathrm{Swap}_{24}\mathrm{Swap}_{35} + \mathrm{Swap}_{24}\mathrm{Swap}_{25} + \mathrm{Swap}_{23}\mathrm{Swap}_{45}$$
$$+ \mathrm{Swap}_{23}\mathrm{Swap}_{34} + \mathrm{Swap}_{23}\mathrm{Swap}_{25} + \mathrm{Swap}_{23}\mathrm{Swap}_{24} + \mathrm{Swap}_{15}\mathrm{Swap}_{34}$$
$$+ \mathrm{Swap}_{15}\mathrm{Swap}_{24} + \mathrm{Swap}_{15}\mathrm{Swap}_{23} - \mathrm{Swap}_{14}\mathrm{Swap}_{25} + \mathrm{Swap}_{14}\mathrm{Swap}_{15}$$
$$- \mathrm{Swap}_{13}\mathrm{Swap}_{25} + \mathrm{Swap}_{13}\mathrm{Swap}_{15} + \mathrm{Swap}_{12}\mathrm{Swap}_{45} + \mathrm{Swap}_{12}\mathrm{Swap}_{35}$$
$$+ \mathrm{Swap}_{12}\mathrm{Swap}_{34} + \mathrm{Swap}_{12}\mathrm{Swap}_{15} + \mathrm{Swap}_{12}\mathrm{Swap}_{14} + \mathrm{Swap}_{12}\mathrm{Swap}_{13}$$
$$- \mathrm{Swap}_{45} - \mathrm{Swap}_{35} - \mathrm{Swap}_{34} - \mathrm{Swap}_{24} - \mathrm{Swap}_{23} - \mathrm{Swap}_{15} - \mathrm{Swap}_{12} + 1$$

$$\mathrm{Swap}_{14}\mathrm{Swap}_{13}\mathrm{Swap}_{12}\mathrm{Swap}_{15} = -\mathrm{Swap}_{12}\mathrm{Swap}_{15}\mathrm{Swap}_{23}\mathrm{Swap}_{24}$$
$$- \mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{15}\mathrm{Swap}_{23} - \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{34}$$
$$- \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{24} - \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{15}$$
$$+ \mathrm{Swap}_{15}\mathrm{Swap}_{23}\mathrm{Swap}_{34} + \mathrm{Swap}_{15}\mathrm{Swap}_{23}\mathrm{Swap}_{24} + \mathrm{Swap}_{14}\mathrm{Swap}_{15}\mathrm{Swap}_{23}$$
$$+ \mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{34} + \mathrm{Swap}_{13}\mathrm{Swap}_{15}\mathrm{Swap}_{24} + \mathrm{Swap}_{13}\mathrm{Swap}_{14}\mathrm{Swap}_{15}$$
$$+ \mathrm{Swap}_{12}\mathrm{Swap}_{15}\mathrm{Swap}_{34} + \mathrm{Swap}_{12}\mathrm{Swap}_{15}\mathrm{Swap}_{24} + \mathrm{Swap}_{12}\mathrm{Swap}_{15}\mathrm{Swap}_{23}$$
$$+ \mathrm{Swap}_{12}\mathrm{Swap}_{14}\mathrm{Swap}_{15} + \mathrm{Swap}_{12}\mathrm{Swap}_{13}\mathrm{Swap}_{15} - \mathrm{Swap}_{15}\mathrm{Swap}_{34}$$
$$- \mathrm{Swap}_{15}\mathrm{Swap}_{24} - \mathrm{Swap}_{15}\mathrm{Swap}_{23} - \mathrm{Swap}_{14}\mathrm{Swap}_{15} - \mathrm{Swap}_{13}\mathrm{Swap}_{15}$$
$$- \mathrm{Swap}_{12}\mathrm{Swap}_{15} + \mathrm{Swap}_{15}$$

B.5. **Gell-Mann matrices of size 4×4.** In the case $d = 4$, the fifteen 4×4 Gell-Mann matrices are

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -\mathfrak{i} & 0 & 0 \\ \mathfrak{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -\mathfrak{i} & 0 \\ 0 & 0 & 0 & 0 \\ \mathfrak{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathfrak{i} & 0 \\ 0 & \mathfrak{i} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \lambda_9 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_{10} = \begin{pmatrix} 0 & 0 & 0 & -\mathfrak{i} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathfrak{i} & 0 & 0 & 0 \end{pmatrix} \quad \lambda_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \lambda_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathfrak{i} \\ 0 & 0 & 0 & 0 \\ 0 & \mathfrak{i} & 0 & 0 \end{pmatrix}$$

$$\lambda_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \lambda_{14} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathfrak{i} \\ 0 & 0 & \mathfrak{i} & 0 \end{pmatrix} \quad \lambda_{15} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

Any product of two such matrices can be expanded in this basis according to a similar formula to (B.2),

$$\text{(B.13)} \qquad \lambda_a \lambda_b = \frac{1}{2}\,\delta_{a,b}\, I + \sum_{c=1}^{15} (d^{a,b,c} + \mathfrak{i} f^{a,b,c})\, \lambda_c,$$

where the structure constants $f^{a,b,c}$ and $d^{a,b,c}$ can be again computed via

$$f^{a,b,c} = -\frac{1}{4}\mathfrak{i}\,\mathrm{tr}(\lambda_a[\lambda_b, \lambda_c]) \quad \text{and} \quad d^{a,b,c} = \frac{1}{4}\,\mathrm{tr}(\lambda_a\{\lambda_b, \lambda_c\}).$$

In this case the nonzero $f^{a,b,c}$ are

$$f^{1,2,3} = 1, \qquad f^{1,5,6} = f^{1,10,11} = f^{3,6,7} = f^{3,11,12} = f^{4,10,13} = f^{6,12,13} = -\frac{1}{2},$$

$$f^{1,4,7} = f^{1,9,12} = f^{2,4,6} = f^{2,5,7} = f^{2,9,11} = f^{2,10,12} = f^{3,4,5} = f^{3,9,10} =$$

$$f^{4,9,14} = f^{5,9,13} = f^{5,10,14} = f^{6,11,14} = f^{7,11,13} = f^{7,12,14} = \frac{1}{2},$$

$$f^{4,5,8} = f^{6,7,8} = \frac{\sqrt{3}}{2}, \qquad f^{8,9,10} = f^{8,11,12} = \frac{1}{2\sqrt{3}},$$

$$f^{8,13,14} = -\frac{1}{\sqrt{3}}, \qquad f^{9,10,15} = f^{11,12,15} = f^{13,14,15} = \sqrt{\frac{2}{3}},$$

and the nonzero $d^{a,b,c}$ are

$$d^{1,1,8} = d^{2,2,8} = d^{3,3,8} = \frac{1}{\sqrt{3}}, \qquad d^{8,8,8} = d^{8,13,13} = d^{8,14,14} = -\frac{1}{\sqrt{3}},$$

$$d^{1,1,15} = d^{2,2,15} = d^{3,3,15} = d^{4,4,15} = d^{5,5,15} = d^{6,6,15} = d^{7,7,15} = d^{8,8,15} = \frac{1}{\sqrt{6}},$$

$$d^{9,9,15} = d^{10,10,15} = d^{11,11,15} = d^{12,12,15} = d^{13,13,15} = d^{14,14,15} = -\frac{1}{\sqrt{6}},$$

$$d^{1,4,6} = d^{1,5,7} = d^{1,9,11} = d^{1,10,12} = d^{2,5,6} = d^{2,10,11} = d^{3,4,4} = d^{3,5,5} = d^{3,9,9} =$$

$$d^{3,10,10} = d^{4,9,13} = d^{4,10,14} = d^{5,10,13} = d^{6,11,13} = d^{6,12,14} = d^{7,12,13} = \frac{1}{2},$$
$$d^{2,4,7} = d^{2,9,12} = d^{3,6,6} = d^{3,7,7} = d^{3,11,11} = d^{3,12,12} = d^{5,9,14} = d^{7,11,14} = -\frac{1}{2},$$
$$d^{4,4,8} = d^{5,5,8} = d^{6,6,8} = d^{7,7,8} = -\frac{1}{2\sqrt{3}}, \qquad d^{15,15,15} = -\sqrt{\frac{2}{3}}$$
$$d^{8,9,9} = d^{8,10,10} = d^{8,11,11} = d^{8,12,12} = \frac{1}{2\sqrt{3}}.$$

Note that the structure constants $f^{a,b,c}$ and $d^{a,b,c}$ with $a, b, c \in \{1, \dots, 8\}$ coincide with the structure constants pertaining to the 3×3 Gell-Mann matrices.

By Proposition 1.9, each swap matrix $\mathrm{Swap}_{ij}^{(4)}$ can be written in terms of the 4×4 Gell-Mann matrices as follows

$$\mathrm{Swap}_{ij}^{(4)} = \frac{1}{4}I + \frac{1}{2}\sum_{a=1}^{15} \lambda_a^i \lambda_a^j. \tag{B.14}$$

B.6. **Linear subspace of $M_{4^n}(\mathbb{C})$ spanned by the products of at most 4 swap matrices.** Again, any two tuples (i, j) and (k, l) are compared w.r.t. the lex ordering. Let $\tilde{\mathcal{B}}_3$ be the set of all products of at most 3 swap matrices that correspond to different permutations in S_n. For fixed $i < j < k < l$ denote by $\mathcal{B}_{ijkl}$ the set consisting of the cubics

$$\begin{gathered}\mathrm{Swap}_{ij}\mathrm{Swap}_{jk}\mathrm{Swap}_{kl},\ \mathrm{Swap}_{ij}\mathrm{Swap}_{jl}\mathrm{Swap}_{kl},\ \mathrm{Swap}_{ik}\mathrm{Swap}_{jk}\mathrm{Swap}_{jl},\\ \mathrm{Swap}_{ik}\mathrm{Swap}_{kl}\mathrm{Swap}_{jl},\ \mathrm{Swap}_{il}\mathrm{Swap}_{jl}\mathrm{Swap}_{jk},\ \mathrm{Swap}_{il}\mathrm{Swap}_{kl}\mathrm{Swap}_{jk}\end{gathered} \tag{B.15}$$

and for fixed $i < j < k < l < m$ denote by $\mathcal{B}_{ijklm}$ the set consisting of the quartics

$$\begin{gathered}\mathrm{Swap}_{ij}\mathrm{Swap}_{jk}\mathrm{Swap}_{kl}\mathrm{Swap}_{lm},\ \mathrm{Swap}_{ij}\mathrm{Swap}_{jk}\mathrm{Swap}_{km}\mathrm{Swap}_{lm},\\ \mathrm{Swap}_{ij}\mathrm{Swap}_{jl}\mathrm{Swap}_{kl}\mathrm{Swap}_{km},\ \mathrm{Swap}_{ij}\mathrm{Swap}_{jl}\mathrm{Swap}_{lm}\mathrm{Swap}_{km},\\ \mathrm{Swap}_{ij}\mathrm{Swap}_{jm}\mathrm{Swap}_{km}\mathrm{Swap}_{kl},\ \mathrm{Swap}_{ij}\mathrm{Swap}_{jm}\mathrm{Swap}_{lm}\mathrm{Swap}_{kl},\\ \mathrm{Swap}_{ik}\mathrm{Swap}_{jk}\mathrm{Swap}_{jl}\mathrm{Swap}_{lm},\ \mathrm{Swap}_{ik}\mathrm{Swap}_{jk}\mathrm{Swap}_{jm}\mathrm{Swap}_{lm},\\ \mathrm{Swap}_{ik}\mathrm{Swap}_{kl}\mathrm{Swap}_{jl}\mathrm{Swap}_{jm},\ \mathrm{Swap}_{ik}\mathrm{Swap}_{kl}\mathrm{Swap}_{lm}\mathrm{Swap}_{jm},\\ \mathrm{Swap}_{ik}\mathrm{Swap}_{km}\mathrm{Swap}_{jm}\mathrm{Swap}_{jl},\ \mathrm{Swap}_{ik}\mathrm{Swap}_{km}\mathrm{Swap}_{lm}\mathrm{Swap}_{jl},\\ \mathrm{Swap}_{il}\mathrm{Swap}_{jl}\mathrm{Swap}_{jk}\mathrm{Swap}_{km},\ \mathrm{Swap}_{il}\mathrm{Swap}_{jl}\mathrm{Swap}_{jm}\mathrm{Swap}_{km},\\ \mathrm{Swap}_{il}\mathrm{Swap}_{kl}\mathrm{Swap}_{jk}\mathrm{Swap}_{jm},\ \mathrm{Swap}_{il}\mathrm{Swap}_{kl}\mathrm{Swap}_{km}\mathrm{Swap}_{jm},\\ \mathrm{Swap}_{il}\mathrm{Swap}_{lm}\mathrm{Swap}_{jm}\mathrm{Swap}_{jk},\ \mathrm{Swap}_{il}\mathrm{Swap}_{lm}\mathrm{Swap}_{km}\mathrm{Swap}_{jk},\\ \mathrm{Swap}_{im}\mathrm{Swap}_{jm}\mathrm{Swap}_{jk}\mathrm{Swap}_{kl},\ \mathrm{Swap}_{im}\mathrm{Swap}_{jm}\mathrm{Swap}_{jl}\mathrm{Swap}_{kl},\\ \mathrm{Swap}_{im}\mathrm{Swap}_{km}\mathrm{Swap}_{jk}\mathrm{Swap}_{jl},\ \mathrm{Swap}_{im}\mathrm{Swap}_{km}\mathrm{Swap}_{kl}\mathrm{Swap}_{jl},\\ \mathrm{Swap}_{im}\mathrm{Swap}_{lm}\mathrm{Swap}_{jl}\mathrm{Swap}_{jk}.\end{gathered} \tag{B.16}$$

Proposition B.5. *The set $\mathcal{B}_4$ consisting of $\tilde{\mathcal{B}}_3$ and the quartics*

$$\mathrm{Swap}_{ij}\mathrm{Swap}_{kl}\mathrm{Swap}_{pq}\mathrm{Swap}_{rs} \qquad \begin{gathered} i < j,\ k < l,\ p < q,\ r < s,\\ (i,j) < (k,l) < (p,q) < (r,s); \end{gathered} \tag{B.17}$$

$$\begin{gathered}\mathrm{Swap}_{ij}\mathrm{Swap}_{jk}\mathrm{Swap}_{pq}\mathrm{Swap}_{rs}\\ \mathrm{Swap}_{ij}\mathrm{Swap}_{ik}\mathrm{Swap}_{pq}\mathrm{Swap}_{rs}\end{gathered} \qquad \begin{gathered} i < j < k,\ p < q, r < s,\\ p, q, r, s \notin \{i, j, k\}, (p,q) < (r,s); \end{gathered} \tag{B.18}$$

$$t \cdot \mathrm{Swap}_{pq}, \quad t \in \mathcal{B}_{ijkl}, \quad i<j<k<l, \; p<q, \; p,q \notin \{i,j,k,l\}; \tag{B.19}$$

$$\begin{aligned} &\mathrm{Swap}_{ij}\mathrm{Swap}_{jk}\mathrm{Swap}_{pq}\mathrm{Swap}_{qr}, \; \mathrm{Swap}_{ij}\mathrm{Swap}_{jk}\mathrm{Swap}_{pq}\mathrm{Swap}_{pr}, \\ &\mathrm{Swap}_{ij}\mathrm{Swap}_{ik}\mathrm{Swap}_{pq}\mathrm{Swap}_{pr}, \quad i<j<k, \; p<q<r, \; i<p, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad \{i,j,k\} \cap \{p,q,r\} = \emptyset; \end{aligned} \tag{B.20}$$

$$t \in \mathcal{B}_{ijklm}, \qquad i<j<k<l<m; \tag{B.21}$$

is a basis of the subspace of $M_n^{Sw_4}(\mathbb{C})$ *of polynomials in the* Swap_{ij} *of degree at most four.*

Proof. For the spanning property of $\mathcal{B}_4$, identify the products of the swap matrices with the corresponding permutations in S_n. Note that the only permutations that can be written as a product of at most four transpositions that we omitted from $\mathcal{B}_4$ are the 5-cycles of the form $(i\,m\,l\,k\,j)$ for $i<j<k<l<m$. But these are in the span of $\mathcal{B}_4$ by the degree-reducing relation (3.1) with $d=4$.

The proof of the linear independence of $\mathcal{B}_4$ again relies on the properties of the 4×4 Gell-Mann matrices presented in Subsection B.5.

Suppose there is a linear dependence among the elements of $\mathcal{B}_4$. Then, using (B.14), express each of the appearing terms w.r.t. the basis (1.9) consisting of different combinations of tensor products of the fifteen 4×4 Gell-Mann matrices.

(a) First, consider the elements in (B.17) and observe that for any choice of $i<j, k<l, p<q, r<s$ with $(i,j)<(k,l)<(p,q)<(r,s)$, the highest order terms in the expansion of $\mathrm{Swap}_{ij}\mathrm{Swap}_{kl}\mathrm{Swap}_{pq}\mathrm{Swap}_{rs}$ are of the form

$$\lambda_a^i\,\lambda_a^j\,\lambda_b^k\,\lambda_b^l\,\lambda_c^p\,\lambda_c^q\,\lambda_d^r\,\lambda_d^s, \qquad a,b,c,d \in \{1,\dots,15\}.$$

By the product formula (B.13), the elements in (B.17) are the only ones that have terms of order eight and more precisely, for any choice of $i<j, k<l, p<q, r<s$ with $(i,j)<(k,l)<(p,q)<(r,s)$, the element $\mathrm{Swap}_{ij}\mathrm{Swap}_{kl}\mathrm{Swap}_{pq}\mathrm{Swap}_{rs}$ has the term $\lambda_1^i\,\lambda_1^j\,\lambda_2^k\,\lambda_2^l\,\lambda_3^p\,\lambda_3^q\lambda_4^r\,\lambda_4^s$, which does not appear in the expansion of any other element of $\mathcal{B}_4$. Hence, the coefficients next to each of the elements in (B.17) have to be zero.

(b) Now the elements in (B.18) are the only ones in $\mathcal{B}_4$ that have terms of order seven (meaning with seven different positions i,j,k,p,q,r,s) in their expansion. For any choice of $i<j<k, p<q, r<s$ with $p,q,r,s \notin \{i,j,k\}, (p,q)<(r,s)$, the highest order terms in the expansion of $\mathrm{Swap}_{ij}\mathrm{Swap}_{jk}\mathrm{Swap}_{pq}\mathrm{Swap}_{rs}$ are of the form

$$\lambda_a^i\,\lambda_a^j\,\lambda_b^j\,\lambda_b^k\,\lambda_c^p\,\lambda_c^q\,\lambda_d^r\,\lambda_d^s, \qquad a,b,c,d \in \{1,\dots,15\},$$

while for $\mathrm{Swap}_{ij}\mathrm{Swap}_{ik}\mathrm{Swap}_{pq}\mathrm{Swap}_{rs}$ they are of the form

$$\lambda_a^i\,\lambda_a^j\,\lambda_b^i\,\lambda_b^k\,\lambda_c^p\,\lambda_c^q\,\lambda_d^r\,\lambda_d^s = \lambda_a^j\,\lambda_a^i\,\lambda_b^i\,\lambda_b^k\,\lambda_c^p\,\lambda_c^q\,\lambda_d^r\,\lambda_d^s, \qquad a,b,c,d \in \{1,\dots,15\}.$$

As noted, the structure constants $f^{a,b,c}$ and $d^{a,b,c}$ with $a,b,c \in \{1,\dots,8\}$ coincide with the structure constants pertaining to the 3×3 Gell-Mann matrices. Hence, similar to part (b) of the proof of Proposition B.2, for any choice of $i<j<k, p<q, r<s$ with $p,q,r,s \notin \{i,j,k\}, (p,q)<(r,s)$, the element $\mathrm{Swap}_{ij}\mathrm{Swap}_{jk}\mathrm{Swap}_{pq}\mathrm{Swap}_{rs}$ has in its expansion

$$\lambda_2^i\,\lambda_2^j\,\lambda_3^j\,\lambda_3^k\,\lambda_5^p\,\lambda_5^q\,\lambda_9^r\,\lambda_9^s + \lambda_1^i\,\lambda_1^j\,\lambda_6^j\,\lambda_6^k\,\lambda_5^p\,\lambda_5^q\,\lambda_9^r\,\lambda_9^s = \mathbf{i}\,\lambda_2^i\,\lambda_1^j\,\lambda_3^k\,\lambda_5^p\,\lambda_5^q\,\lambda_9^r\,\lambda_9^s + \frac{1}{2}\lambda_1^i\,\lambda_4^j\,\lambda_6^k\,\lambda_5^p\,\lambda_5^q\,\lambda_9^r\,\lambda_9^s.$$

But $\text{Swap}_{ij}\text{Swap}_{ik}\text{Swap}_{pq}$ has in its expansion

$$\lambda_1^i\,\lambda_1^j\,\lambda_3^i\,\lambda_3^k\,\lambda_5^p\,\lambda_5^q\,\lambda_9^r\,\lambda_9^s+\lambda_1^i\,\lambda_1^j\,\lambda_6^i\,\lambda_6^k\,\lambda_5^p\,\lambda_5^q\,\lambda_9^r\,\lambda_9^s = -\mathbf{i}\,\lambda_2^i\,\lambda_1^j\,\lambda_3^k\,\lambda_5^p\,\lambda_5^q\,\lambda_9^r\,\lambda_9^s+\frac{1}{2}\lambda_1^i\,\lambda_4^j\,\lambda_6^k\,\lambda_5^p\,\lambda_5^q\,\lambda_9^r\,\lambda_9^s.$$

By the same argument as in part (b) of the proof of Proposition B.2, all the coefficients next to the elements in (B.18) are zero.

(c) Now the elements in (B.19) and (B.20) are the only ones with terms of order six (i.e., with six different positions denoted by either i, j, k, l, p, q or i, j, k, p, q, r) in their expansion.

For the elements in (B.19), given any choice of $i < j < k < l$, $p < q$ with $p, q \notin \{i, j, k, l\}$, the highest order terms are of the form

$$\boldsymbol{\lambda}\cdot\lambda_a^p\,\lambda_a^q,$$

where $\boldsymbol{\lambda}$ is a highest order term of an element of $\mathcal{B}_{ijkl}$ as in the proof of Proposition B.2. For the elements in (B.20), given any choice of $i < j < k, p < q < r, i < p$ with $\{i, j, k\} \cap \{p, q, r\} = \emptyset$, the highest order terms of the three appearing types are

$$\begin{aligned}
\text{Swap}_{ij}\text{Swap}_{jk}\text{Swap}_{pq}\text{Swap}_{qr}: &\quad \lambda_a^i\,\lambda_a^j\,\lambda_b^j\,\lambda_b^k\,\lambda_c^p\,\lambda_c^q\,\lambda_d^q\,\lambda_d^r\\
\text{Swap}_{ij}\text{Swap}_{jk}\text{Swap}_{pq}\text{Swap}_{pr}: &\quad \lambda_a^i\,\lambda_a^j\,\lambda_b^j\,\lambda_b^k\,\lambda_c^p\,\lambda_c^q\,\lambda_d^p\,\lambda_d^r\\
\text{Swap}_{ij}\text{Swap}_{ik}\text{Swap}_{pq}\text{Swap}_{pr}: &\quad \lambda_a^i\,\lambda_a^j\,\lambda_b^i\,\lambda_b^k\,\lambda_c^p\,\lambda_c^q\,\lambda_d^p\,\lambda_d^r.
\end{aligned}$$

First, consider (B.20). For fixed $i < j < k, p < q < r, i < p$ with $\{i, j, k\} \cap \{p, q, r\} = \emptyset$, denote the coefficients next to the elements

$$\text{Swap}_{ij}\text{Swap}_{jk}\text{Swap}_{pq}\text{Swap}_{qr}, \text{Swap}_{ij}\text{Swap}_{jk}\text{Swap}_{pq}\text{Swap}_{pr}, \text{Swap}_{ij}\text{Swap}_{ik}\text{Swap}_{pq}\text{Swap}_{pr}$$

by β_1, β_2 and β_3 respectively. Clearly, these are the only elements in (B.20) whose highest order terms involve precisely the positions i, j, k, p, q, r. So comparing the coefficients next to the basis elements $\lambda_2^i\,\lambda_{10}^j\,\lambda_{11}^k\,\lambda_1^p\,\lambda_4^q\,\lambda_7^r$, $\lambda_2^i\,\lambda_9^j\,\lambda_{11}^k\,\lambda_1^p\,\lambda_5^q\,\lambda_7^r$ and $\lambda_2^i\,\lambda_9^j\,\lambda_{11}^k\,\lambda_1^p\,\lambda_4^q\,\lambda_7^r$ give the following equations

$$\begin{aligned}
\lambda_2^i\,\lambda_{10}^j\,\lambda_{11}^k\,\lambda_1^p\,\lambda_4^q\,\lambda_7^r: &\quad -\beta_1 - \beta_2 + \beta_3 = 0,\\
\lambda_2^i\,\lambda_9^j\,\lambda_{11}^k\,\lambda_1^p\,\lambda_5^q\,\lambda_7^r: &\quad -\beta_1 + \beta_2 + \beta_3 = 0,\\
\lambda_2^i\,\lambda_9^j\,\lambda_{11}^k\,\lambda_1^p\,\lambda_4^q\,\lambda_7^r: &\quad -\beta_1 + \beta_2 - \beta_3 = 0.
\end{aligned}$$

The above system has a unique solution $\beta_1 = \beta_2 = \beta_3 = 0$. Note that each of the highest order terms of the elements in (B.19) necessarily has one of the Gell-Mann matrices λ repeated twice. So the coefficients next to the basis elements $\lambda_2^i\,\lambda_{10}^j\,\lambda_{11}^k\,\lambda_1^p\,\lambda_4^q\,\lambda_7^r$, $\lambda_2^i\,\lambda_9^j\,\lambda_{11}^k\,\lambda_1^p\,\lambda_5^q\,\lambda_7^r$ and $\lambda_2^i\,\lambda_9^j\,\lambda_{11}^k\,\lambda_1^p\,\lambda_4^q\,\lambda_7^r$ in the expansion of the elements in (B.19) are zero. We conclude that the coefficients next to the elements

$$\text{Swap}_{ij}\text{Swap}_{jk}\text{Swap}_{pq}\text{Swap}_{qr}, \text{Swap}_{ij}\text{Swap}_{jk}\text{Swap}_{pq}\text{Swap}_{pr}, \text{Swap}_{ij}\text{Swap}_{ik}\text{Swap}_{pq}\text{Swap}_{pr},$$

are zero and by analogy, the coefficients next to all the elements in (B.20) are zero.

Now consider (B.19). For fixed $i < j < k < l$, $p < q$ with $p, q \notin \{i, j, k, l\}$, denote the coefficients next to the elements

$$\begin{gathered}
\text{Swap}_{ij}\text{Swap}_{jk}\text{Swap}_{kl}\text{Swap}_{pq},\ \text{Swap}_{ij}\text{Swap}_{jl}\text{Swap}_{kl}\text{Swap}_{pq},\ \text{Swap}_{ik}\text{Swap}_{jk}\text{Swap}_{jl}\text{Swap}_{pq},\\
\text{Swap}_{ik}\text{Swap}_{kl}\text{Swap}_{jl}\text{Swap}_{pq},\ \text{Swap}_{il}\text{Swap}_{jl}\text{Swap}_{jk}\text{Swap}_{pq},\ \text{Swap}_{il}\text{Swap}_{kl}\text{Swap}_{jk}\text{Swap}_{pq}
\end{gathered}$$

by $\alpha_1, \ldots, \alpha_6$ respectively. Clearly, these are the only elements in (B.19) whose highest order terms involve precisely the positions i, j, k, l, p, q. So comparing the coefficients next to several basis elements of the form $\lambda_a^i\,\lambda_b^j\,\lambda_c^k\,\lambda_d^l\,\lambda_e^p\,\lambda_e^q$ gives the following equations

$$\begin{aligned}
\lambda_{11}^i\,\lambda_1^j\,\lambda_5^k\,\lambda_{13}^l\,\lambda_2^p\,\lambda_2^q: &\quad \beta_1-\beta_6=0,\\
\lambda_{11}^i\,\lambda_1^j\,\lambda_{13}^k\,\lambda_5^l\,\lambda_2^p\,\lambda_2^q: &\quad \beta_2-\beta_5=0,\\
\lambda_{11}^i\,\lambda_5^j\,\lambda_1^k\,\lambda_{13}^l\,\lambda_2^p\,\lambda_2^q: &\quad \beta_3-\beta_4=0,\\
\lambda_{11}^i\,\lambda_1^j\,\lambda_3^k\,\lambda_9^l\,\lambda_2^p\,\lambda_2^q: &\quad \beta_1-\beta_3-\beta_4=0,\\
\lambda_{11}^i\,\lambda_3^j\,\lambda_1^k\,\lambda_9^l\,\lambda_2^p\,\lambda_2^q: &\quad -\beta_1+\beta_3+\beta_4-\beta_6=0,\\
\lambda_{11}^i\,\lambda_9^j\,\lambda_1^k\,\lambda_3^l\,\lambda_2^p\,\lambda_2^q: &\quad -\beta_1+\beta_2+\beta_5-\beta_6=0.
\end{aligned}\tag{B.22}$$

The above system has a unique solution $\beta_1=\cdots\beta_6=0$. Hence, by analogy, the coefficients next to all of the elements in (B.19) are zero.

(d) The quartics in (B.21) are now the only ones in $\mathcal{B}_4$ that have terms of degree five. For fixed $i<j<k<l<m$ denote the coefficients next to the quartics in (B.16) by $\gamma_1,\dots,\gamma_{23}$ respectively. Clearly, these are the only elements in (B.21) whose highest order terms involve precisely the positions i,j,k,l,m. Similar to before, we now compare the coefficients next to several basis elements of the form $\lambda_a^i\,\lambda_b^j\,\lambda_c^k\,\lambda_d^l\,\lambda_e^m$ to get a system of equations. By symmetry note that if $\lambda_a^i\,\lambda_b^j\,\lambda_c^k\,\lambda_d^l\,\lambda_e^m$ is, e.g., a term in the expansion of $\mathrm{Swap}_{ij}\mathrm{Swap}_{jk}\mathrm{Swap}_{kl}\mathrm{Swap}_{lm}$ and σ is a permutation of the positions i,j,k,l,m, then $\lambda_a^{\sigma(i)}\,\lambda_b^{\sigma(j)}\,\lambda_c^{\sigma(k)}\,\lambda_d^{\sigma(l)}\,\lambda_e^{\sigma(m)}$ is a term in the expansion of

$$\mathrm{Swap}_{\sigma(i),\sigma(j)}\mathrm{Swap}_{\sigma(j),\sigma(k)}\mathrm{Swap}_{\sigma(k),\sigma(l)}\mathrm{Swap}_{\sigma(l),\sigma(m)}.$$

By this observation it is easy to quickly deduce several equations, e.g.,

$$\begin{aligned}
\lambda_{11}^i\,\lambda_9^j\,\lambda_4^k\,\lambda_6^l\,\lambda_3^m: &\ -\gamma_1+\gamma_{21}+\gamma_4=0,\\
\lambda_{11}^i\,\lambda_9^j\,\lambda_4^k\,\lambda_3^l\,\lambda_6^m: &\ -\gamma_2+\gamma_{22}-\gamma_{23}+\gamma_3=0,\\
\lambda_{11}^i\,\lambda_9^j\,\lambda_6^k\,\lambda_4^l\,\lambda_3^m: &\ -\gamma_3+\gamma_{22}-\gamma_{19}+\gamma_6=0,\\
\lambda_{11}^i\,\lambda_9^j\,\lambda_6^k\,\lambda_3^l\,\lambda_4^m: &\ -\gamma_4+\gamma_{21}-\gamma_{20}+\gamma_5=0,\\
\lambda_{11}^i\,\lambda_9^j\,\lambda_3^k\,\lambda_4^l\,\lambda_6^m: &\ -\gamma_5+\gamma_{20}+\gamma_1=0.
\end{aligned}\tag{B.23}$$

The remaining 19 equations are computed by analogy. To apply this technique correctly it is important to keep in mind that the equation given by $\lambda_{11}^i\,\lambda_9^j\,\lambda_4^k\,\lambda_6^l\,\lambda_3^m$ is in fact $-\gamma_1+\gamma_{21}-\gamma_{24}+\gamma_4=0$, where $\gamma_{24}=0$ is the coefficient corresponding to $\mathrm{Swap}_{im}\mathrm{Swap}_{lm}\mathrm{Swap}_{kl}\mathrm{Swap}_{jk}$ (that we excluded from the basis, but that we need to keep in mind to calculate the following equations correctly). Combining all the 24 equations (B.23) with the ones obtained by considering the terms

$$\lambda_{11}^i\,\lambda_1^j\,\lambda_{15}^k\,\lambda_3^l\,\lambda_9^m,\ \lambda_{11}^i\,\lambda_1^j\,\lambda_{15}^k\,\lambda_9^l\,\lambda_3^m,\ \lambda_{11}^i\,\lambda_1^j\,\lambda_3^k\,\lambda_{15}^l\,\lambda_9^m,$$
$$\lambda_{11}^i\,\lambda_1^j\,\lambda_9^k\,\lambda_3^l\,\lambda_{15}^m,\ \lambda_{11}^i\,\lambda_{15}^j\,\lambda_1^k\,\lambda_9^l\,\lambda_3^m,\ \lambda_{11}^i\,\lambda_9^j\,\lambda_{15}^k\,\lambda_1^l\,\lambda_3^m,$$

we get a system with unique solution $\gamma_1=\cdots=\gamma_{23}=0$. Hence, by analogy, the coefficients next to all of the quartics in (B.21) are zero as well.

(e) Now the cubics (B.15) are the only ones with terms of degree four in their expansion. But comparing the coefficients next to the basis elements

$$\lambda_{11}^i\,\lambda_1^j\,\lambda_5^k\,\lambda_{13}^l,\ \lambda_{11}^i\,\lambda_1^j\,\lambda_{13}^k\,\lambda_5^l,\ \lambda_{11}^i\,\lambda_5^j\,\lambda_1^k\,\lambda_{13}^l,\ \lambda_{11}^i\,\lambda_1^j\,\lambda_3^k\,\lambda_9^l,\ \lambda_{11}^i\,\lambda_3^j\,\lambda_1^k\,\lambda_9^l,\ \lambda_{11}^i\,\lambda_9^j\,\lambda_1^k\,\lambda_3^l$$

gives back the system (B.22), which implies that the coefficients next to all of the cubics are zero as well. We are left with a linear dependence involving terms of degree at most two, which contradicts linear independence of $\mathcal{B}_2$ as shown in Proposition B.1. ■

Appendix C. Explicit eigenvalue computation for clique Hamiltonians of general d-row partitions

Here, we give an alternative and elementary method to compute the character value $\chi_\lambda((i\,j))$ of a transposition $(i\,j)$ using the Murnaghan-Nakayama rule [Pro07, Section 9.9.1]. Using formula (6.3), we then again compute, for any partition $\lambda \vdash n$ with d rows, the eigenvalue η_λ from Lemma 6.4.

C.1. **The Murnaghan-Nakayama rule.** To compute the value of the character χ_λ at the conjugacy class of transpositions we use the non-recursive version of the Murnaghan-Nakayama rule. It states that

$$\chi_\lambda\big((i\ j)\big) = \sum_T (-1)^{\mathrm{ht}(T)}, \tag{C.1}$$

where the sum runs over all tableaux T of shape λ that satisfy:

- the boxes of T are filled with numbers $1, 2, \ldots, n-1$ such that 1 appears twice and all the others appear once,
- the numbers in every row and column are weakly increasing.

Here $\mathrm{ht}(T)$ is one if both $1'$s are in the first column and it is zero if they are in the first row.

Clearly, the set of all such tableaux is in bijection with the set of all standard Young tableaux of shape λ. This means that

$$\begin{aligned}\chi_\lambda\big((i\ j)\big) = \#&(\text{standard Young tableaux with 1 and 2 in the first row})-\\ \#&(\text{standard Young tableaux with 1 and 2 in the first column}).\end{aligned}$$

It is easy to see that the standard Young Tableaux with 1 and 2 in the first row are in bijection with the standard Young tableaux of the shape that we get by removing the first two boxes from the first row of λ. Similarly, the standard Young Tableaux with 1 and 2 in the first column are in bijection with the standard Young tableaux of the shape that we get by removing the first two boxes from the first column of λ. To count these we use the hook-length formula for skew shaped Young tableaux [Nar14, MPP18].

For a partition μ denote by $[\mu]$ the diagram (i.e., tableaux without numbers) of shape μ. If $[\mu]$ is the diagram that we cut out of $[\lambda]$, then denote the resulting skew shaped diagram by $[\lambda/\mu]$ and the number of all standard Young tableaux of shape λ/μ by $f^{\lambda/\mu}$. For a box $(i,j) \in [\mu]$ such that the boxes $(i+1,j), (i,j+1), (i+1,j+1) \in [\lambda]$ are not in $[\mu]$, we say that an **excited move** with respect to λ is the replacement of $[\mu]$ by

$$\Big([\mu]\backslash\{(i\ j)\}\Big) \cup \{(i+1\ j+1)\}.$$

An **excited diagram** of shape λ/μ is a diagram contained in $[\lambda]$ that can be obtained from $[\mu]$ with a series of excited moves. Denote by $\mathcal{E}(\lambda/\mu)$ the set of all excited diagrams of shape λ/μ (here $\mathcal{E}(\lambda/\mu)$ is empty unless $[\mu] \subseteq [\lambda]$) (see Example C.1).

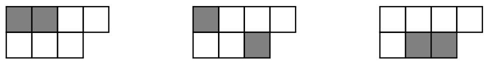
The hook-length formula for skew shaped tableaux [MPP18, Theorem 1.2] states that

$$f^{\lambda/\mu} = |\lambda/\mu|! \sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{ij} \frac{1}{\mathrm{hook}_{\lambda/\mu}(i,j)}. \tag{C.2}$$

In our case, $\chi_\lambda((i,j)) = f^{\lambda/\mu}$ where $\mu = (1,1)$ is the two-row partition of two. Hence, if (i_1, j_1) and (i_2, j_2) are the two distinguished squares in an excited diagram D of shape λ/μ, the summand in (C.2) pertaining to D can be expressed as

$$\frac{\chi_\lambda(e)}{n!}\,\mathrm{hook}_{\lambda/\mu}(i_1,j_1)\,\mathrm{hook}_{\lambda/\mu}(i_2,j_2). \tag{C.3}$$

Example C.1. For $n = 7$ let $\lambda = (4,3)$ and $\mu = (2)$. Then there are three excited diagrams of shape λ/μ,



C.2. **Clique eigenvalue computation.** Next we present an alternative method to prove Proposition 6.7, which we restate below.

Proposition 6.7. *Let η_λ be as in Lemma 6.4. For any $\lambda \vdash n$ with rows $\lambda_1 \geq \cdots \geq \lambda_d$,*

$$\eta_\lambda = n^2 + \frac{d(d-1)(2d-1)}{6} - \sum_{k=1}^{d} \big(\lambda_k - (k-1)\big)^2. \tag{C.4}$$

Sketch of proof. Let $\lambda \vdash n$ be a partition with at most d rows $\lambda_1 \geq \cdots \geq \lambda_d \geq 0$. To calculate η_λ through the formula (6.3) we first compute the value $\chi_\lambda((i\,j))$ of the character χ_λ at the conjugacy class of transpositions. For that we use the Murnaghan-Nakayama rule as presented in Subsection C.1.

First consider the excited diagrams of shape λ/μ with $\mu = (2)$. We only tackle the most general case with $\lambda_d \geq d+1$ (i.e., the case that gives the most excited diagrams). By a similar reasoning as before, the box $(1,1)$, can be moved only after the box $(1,2)$ had already been moved. If $(1,1)$ is, say, in position (k,k) for $k = 1,\dots,d$, this means that $(1,2)$ must have been moved to one of the $d-k+1$ positions $(k,k+1),\dots,(d,d+1)$. So if $(1,1)$ is moved to (k,k) and $(1,2)$ is moved to $(j,j+1)$ for some $j = k,\dots,d$, the contribution to (C.2) of this excited diagram computed via (C.3) is

$$\frac{\chi_\lambda(e)}{n!}\,\mathrm{hook}_{\lambda/(2)}(k,k)\,\mathrm{hook}_{\lambda/(2)}(j,j+1) = \frac{\chi_\lambda(e)}{n!}\,(\lambda_k - (k-1) + d - k)(\lambda_j - j + d - j).$$

Hence,

$$f^{\lambda/(2)} = \frac{\chi_\lambda(e)}{n(n-1)} \sum_{k=1}^{d}\sum_{j=k}^{d} (\lambda_k - 2k + d + 1)(\lambda_j - 2j + d).$$

For the excited diagrams of shape λ/μ with $\mu = (1,1)$, we again only consider the case with $\lambda_d \geq d-1$, which gives the most excited diagrams. In this case, if the box $(1,1)$ is moved to position (k,k) for some $k = 1,\dots,d-1$, the box $(2,1)$ must have been moved to one of the $d-k$ positions $(k+1,k),\dots,(d,d-1)$. So if $(1,1)$ is moved to (k,k) and $(2,1)$ is moved to $(j+1,j)$ for some $j = k,\dots,d-1$, the contribution to (C.2) of this excited diagram computed via (C.3) is now

$$\begin{aligned}&\frac{\chi_\lambda(e)}{n!}\,\mathrm{hook}_{\lambda/(1,1)}(k,k)\,\mathrm{hook}_{\lambda/(1,1)}(j+1,j)\\ &= \frac{\chi_\lambda(e)}{n!}\,(\lambda_k - (k-1) + d - k)(\lambda_j - (j-2) + d - j).\end{aligned}$$

Hence,

$$f^{\lambda/(1,1)} = \frac{\chi_\lambda(e)}{n(n-1)} \sum_{k=1}^{d-1} \sum_{j=k+1}^{d} (\lambda_k - 2k + d + 1)(\lambda_j - 2j + d + 2).$$

Putting all together we get

$$\begin{aligned} \eta_\lambda &= 2\binom{n}{2}\left(1 - \frac{\chi_\lambda\big((i\ j)\big)}{\chi_\lambda(e)}\right) = 2\binom{n}{2}\left(1 - \frac{f^{\lambda/(2)} - f^{\lambda/(1,1)}}{\chi_\lambda(e)}\right) \\ &= n(n-1) - \sum_{k=1}^{d}\sum_{j=k}^{d}(\lambda_k - 2k + d + 1)(\lambda_j - 2j + d) \\ &+ \sum_{k=1}^{d-1} \sum_{j=k+1}^{d} (\lambda_k - 2k + d + 1)(\lambda_j - 2j + d + 2) \\ &= n^2 - 2n - \sum_{k=1}^{d}(\lambda_k^2 - 2k\lambda_k) \\ &= n^2 + \frac{d(d-1)(2d-1)}{6} - \sum_{k=1}^{d} \big(\lambda_k - (k-1)\big)^2. \end{aligned}$$

■

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