ON DOMAINS OF NONCOMMUTATIVE RATIONAL FUNCTIONS

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Abstract. In this paper the stable extended domain of a noncommutative rational function is introduced and it is shown that it can be completely described by a monic linear pencil from the minimal realization of the function. This result amends the singularities theorem of Kalyuzhnyi-Verbovetskyi and Vinnikov. Furthermore, for noncommutative rational functions which are regular at a scalar point it is proved that their domains and stable extended domains coincide.

1. Introduction

Noncommutative rational fractions are the elements of the universal skew field of a free algebra [Coh95]. While this skew field can be constructed in various ways [Ami66, Le74, Li00], it can also be defined through evaluations of formal rational expressions on tuples of matrices [K-VV12]. This interpretation gives rise to prominent applications of noncommutative rational functions in free analysis [K-VV14, AM15], free real algebraic geometry [HMV06, OHMP09, BPT13, HM14] and control theory [BGM05, BH10]. The consideration of matrix evaluations naturally leads to the introduction of the *domain* of a noncommutative rational function. However, at first sight this notion seems intangible: since a noncommutative rational function is an equivalence class of formal rational expressions, its domain is defined as the union of the formal domains of all its representatives; see Subsection 2.1 for precise definition. Therefore new variants of domains emerged: *extended domains* [K-VV09, K-VV12] and *analytic* or *limit domains* [HMV06, HM14]. Both of these notions are related to generic evaluations of noncommutative rational functions and can thus be described using a single representative of a function.

The main important breakthrough in characterizing domains was done by Dmitry Kaliuzhnyi-Verbovetskyi and Victor Vinnikov in [K-VV09]. Perceptively combining linear systems realizations from control theory and difference-differential operators from free analysis they seemingly proved that the extended domain of a noncommutative rational function $\mathbb{r}$ that is regular at the origin coincides with the invertibility set of the monic linear pencil from a minimal realization of $\mathbb{r}$; see [K-VV09, Theorem 3.1]. While minimal realizations of noncommutative rational functions can be effectively computed [BGM05, BR11], monic linear pencils are key tools in matrix theory and are well-explored through control theory [BEFB94], algebraic geometry [Dol12] and optimization [WSV12]. Therefore the result of Kaliuzhnyi-Verbovetskyi and Vinnikov proved to be of great importance in free real algebraic geometry and free function theory [HMPV09, BH10, MS13, HM14, KV+, KPV+]. Alas, there is a gap in its proof and its conclusion does not hold.

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The reason behind is the fact that the extended domain of a noncommutative rational function is in general not closed under direct sums; a concrete instance when this occurs is given in Example 2.1.

The main results of this paper adjust and improve [K-VV09, Theorem 3.1]. First we recall domains and extended domains in Subsection 2.1 and give the necessary facts about realizations in Subsection 2.2. Then we define the *stable extended domain* of a noncommutative rational function (Definition 3.1). In Proposition 3.3 it is shown that the stable extended domain is always closed under direct sums, which is in contrast with the extended domain. In Theorem 3.5 we prove that the following variant of [K-VV09, Theorem 3.1] holds.

If a noncommutative rational function $\mathbb{r}$ *is regular at the origin, then its stable extended domain is equal to the invertibility set of the monic linear pencil from the minimal realization of* $\mathbb{r}$*.*

This statement is then extended to noncommutative rational functions that are regular at some scalar point in Corollary 3.7. Moreover, for such functions we are able to completely describe their domains due to the following result.

If a noncommutative rational function is regular at some scalar point, then its domain and its stable extended domain coincide.

See Theorem 3.10 for the proof. Finally, in Example 3.13 we present a noncommutative rational function whose domain is strictly larger than the domain of any of its representatives.

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2. Domains, extended domains and linear pencils

In this section we first recall the definition of noncommutative rational functions, their (extended) domains and the basics of realization theory. Then we present a counterexample to [K-VV09, Theorem 3.1] and explain why this phenomenon occurs.

2.1. **Skew field of noncommutative rational functions.** Throughout the paper let $\mathbb{k}$ be a field of characteristic 0. Let $\boldsymbol{x} = (x_1, \dots, x_g)$ be a tuple of freely noncommuting variables. By $<\boldsymbol{x}>$ and $\mathbb{k}<\boldsymbol{x}>$ we denote the free monoid and the free unital $\mathbb{k}$-algebra, respectively, generated by $\boldsymbol{x}$. Elements of $<\boldsymbol{x}>$ and $\mathbb{k}<\boldsymbol{x}>$ are called *words* and *noncommutative (nc) polynomials*, respectively.

We introduce noncommutative rational functions using matrix evaluations of formal rational expressions following [HMV06, K-VV12]. For their ring-theoretic origins see [Ami66, Coh95]. **Noncommutative (nc) rational expressions** are syntactically valid combinations of elements in $\mathbb{k}$, variables in $\boldsymbol{x}$, arithmetic operations $+, \cdot, ^{-1}$ and parentheses $(,)$. For example, $(x_3 + x_2 x_3^{-1} x_1)^{-1} + 1$, $x_1^{-1} x_2^{-1} - (x_2 x_1)^{-1}$ and $(1 - x_1^{-1} x_1)^{-1}$ are nc rational expressions.

Fix $n \in \mathbb{N}$. Given a nc rational expression r and $X \in \mathrm{M}_n(\mathbb{k})^g$, the evaluation $r(X)$ is defined in the obvious way if all inverses appearing in r exist at X. Let $\operatorname{dom}_n r$ be the set of all $X \in \mathrm{M}_n(\mathbb{k})^g$ such that r is regular at X. Note that $\operatorname{dom}_n r$ is Zariski open in

$\mathrm{M}_n(\Bbbk)^g$. The set

$$\operatorname{dom} r = \bigcup_{n\in\mathbb{N}} \operatorname{dom}_n r$$

is called **the domain of** r. We say that a nc rational expression r is **non-degenerate** if $\operatorname{dom} r \neq \emptyset$. Let $\mathcal{R}_\Bbbk(\boldsymbol{x})$ denote the set of all non-degenerate expressions and on it we define an equivalence relation $r_1 \sim r_2$ if and only if $r_1(X) = r_2(X)$ for all $X \in \operatorname{dom} r_1 \cap \operatorname{dom} r_2$. The equivalence classes with respect to this relation are called **noncommutative (nc) rational functions**. By [K-VV12, Proposition 2.1] they form a skew field denoted $\Bbbk\mathopen{<\!\!\!\{}\boldsymbol{x}\mathclose{\}\!\!\!>}$, which is the universal skew field of fractions of $\Bbbk\langle\boldsymbol{x}\rangle$; see [Coh95, Section 4.5] for an exposition on universal skew fields. The equivalence class of a nc rational expression $r \in \mathcal{R}_\Bbbk(\boldsymbol{x})$ is written as $\mathbb{r} \in \Bbbk\mathopen{<\!\!\!\{}\boldsymbol{x}\mathclose{\}\!\!\!>}$. Denote

$$\operatorname{dom}_n \mathbb{r} = \bigcup_{r\in\mathbb{r}} \operatorname{dom}_n r, \qquad \operatorname{dom} \mathbb{r} = \bigcup_{n\in\mathbb{N}} \operatorname{dom}_n \mathbb{r}$$

and call $\operatorname{dom}\mathbb{r}$ **the domain of** $\mathbb{r}$.

Again fix $n \in \mathbb{N}$ and let $\Xi = (\Xi_1, \dots, \Xi_g)$ be the $n \times n$ **generic matrices**, i.e., the matrices whose entries are independent commuting variables. If $\operatorname{dom}_n r \neq \emptyset$ for $r \in \mathcal{R}_\Bbbk(\boldsymbol{x})$, then $r[n] := r(\Xi)$ is a $n \times n$ matrix of commutative rational functions in gn^2 variables. If $r_1, r_2 \in \mathcal{R}_\Bbbk(\boldsymbol{x})$ and $\operatorname{dom}_n r_1 \neq \emptyset \neq \operatorname{dom}_n r_2$, then $r_1 \sim r_2$ clearly implies $r_1[n] = r_2[n]$. For $\mathbb{r} \in \Bbbk\mathopen{<\!\!\!\{}\boldsymbol{x}\mathclose{\}\!\!\!>}$ with $\operatorname{dom}_n \mathbb{r} \neq \emptyset$ we can thus set $\mathbb{r}[n] := r[n]$ for arbitrary $r \in \mathbb{r}$ with $\operatorname{dom}_n r \neq \emptyset$. Since the ring of commutative polynomials is a unique factorization ring, every commutative rational function is a quotient of two (up to a scalar multiple) unique coprime polynomials and therefore has a well-defined domain, namely the complement of the zero set of its denominator. Let $\operatorname{edom}_n \mathbb{r} \subseteq \mathrm{M}_n(\Bbbk)^g$ be the intersection of the domains of entries in $\mathbb{r}[n]$ if $\operatorname{dom}_n \mathbb{r} \neq \emptyset$; otherwise set $\operatorname{edom}_n \mathbb{r} = \emptyset$. Note that $\operatorname{edom}_n \mathbb{r}$ is Zariski open in $\mathrm{M}_n(\Bbbk)^g$. The set

$$\operatorname{edom} \mathbb{r} = \bigcup_{n\in\mathbb{N}} \operatorname{edom}_n \mathbb{r}$$

is called **the extended domain of** $\mathbb{r}$; see [K-VV09, Subsection 2.1].

2.2. **Realization theory.** A powerful tool for operating with nc rational functions regular at the origin are realizations, coming from control theory [BGM05] and automata theory [BR11]. Let $\mathbb{r} \in \Bbbk\mathopen{<\!\!\!\{}\boldsymbol{x}\mathclose{\}\!\!\!>}$ and assume that $0 \in \operatorname{dom}\mathbb{r}$. Then there exist $d \in \mathbb{N}$, vectors $\mathbf{b}, \mathbf{c} \in \Bbbk^{d\times 1}$ and a **monic linear pencil** $L = I - \sum_j A_j x_j$ with $A_1, \dots, A_g \in \Bbbk^{d\times d}$ such that

$$\mathbb{r} = \mathbf{c}^t L^{-1} \mathbf{b} \tag{2.1}$$

holds in $\Bbbk\mathopen{<\!\!\!\{}\boldsymbol{x}\mathclose{\}\!\!\!>}$. The tuple $(\mathbf{c}, L, \mathbf{b})$ is called a **(recognizable series) realization** of $\mathbb{r}$ of size d; cf. [BR11]. In general, $\mathbb{r}$ admits various realizations. The ones whose size is minimal among all realizations of $\mathbb{r}$ are called **minimal**. It is well known (see e.g. [BR11, Theorem 2.4]) that minimal realizations are unique up to similarity: if $(\mathbf{c}, L, \mathbf{b})$ and $(\mathbf{c}', L', \mathbf{b}')$ are minimal realizations of $\mathbb{r}$ of size d, then there exists $P \in \mathrm{GL}_d(\Bbbk)$ such that $\mathbf{c}' = P^{-t}\mathbf{c}$, $\mathbf{b}' = P\mathbf{b}$ and $A'_j = PA_jP^{-1}$ for $1 \le j \le g$.

For a monic pencil L as above and $X \in \mathrm{M}_n(\Bbbk)^g$ we write $L(X) = I \otimes I - \sum_j A_j \otimes X_j$. For $\mathbb{r} \in \Bbbk\mathopen{<\!\!\!\{}\boldsymbol{x}\mathclose{\}\!\!\!>}$ with $0 \in \operatorname{dom}\mathbb{r}$ define

$$\mathcal{D}_n(\mathbb{r}; 0) = \{X \in \mathrm{M}_n(\Bbbk)^g \colon \det L(X) \neq 0\}, \qquad \mathcal{D}(\mathbb{r}; 0) = \bigcup_n \mathcal{D}_n(\mathbb{r}; 0),$$

where $(\mathbf{c}, L, \mathbf{b})$ is a minimal realization of $\mathbb{r}$. Note that $\mathcal{D}(\mathbb{r}; 0)$ is independent of the choice of a minimal realization because of its uniqueness up to similarity. This set is also called the *invertibility set* of L [HM14, Subsection 1.1].

More generally, if $\alpha \in \operatorname{dom}_1 \mathbb{r}$, then $\mathbb{r}_\alpha(x) = \mathbb{r}(x + \alpha)$ is a nc rational function regular at 0 and we define

$$\mathcal{D}_n(\mathbb{r}; \alpha) = I_n \alpha + \mathcal{D}_n(\mathbb{r}_\alpha; 0), \qquad \mathcal{D}(\mathbb{r}; \alpha) = \bigcup_n \mathcal{D}_n(\mathbb{r}; \alpha).$$

2.3. **Ill-behavior of the extended domain.** Obviously we have $\operatorname{dom} \mathbb{r} \subseteq \operatorname{edom} \mathbb{r}$. Also, if $\alpha \in \operatorname{dom}_1 \mathbb{r}$, then $\mathcal{D}(\mathbb{r}; \alpha) \subseteq \operatorname{edom} \mathbb{r}$. In the following example we show that $\mathcal{D}(\mathbb{r}; 0)$ is not necessarily equal to $\operatorname{edom} \mathbb{r}$, thus presenting a counterexample to [K-VV09, Theorem 3.1].

Example 2.1. Let $g = 2$ and $r = (1 - x_1)x_2(1 - x_1)^{-1}$. Then $\mathbb{r}$ admits a minimal realization

$$\begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -x_2 \\ x_1 & 1 & -x_2 \\ 0 & 0 & 1 - x_1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

so $D_1(\mathbb{r}; 0) = \{(x_1, x_2) \colon x_1 \neq 1\}$. On the other hand, if ξ_1, ξ_2 are commuting independent variables, then $(1 - \xi_1)\xi_2(1 - \xi_1)^{-1} = \xi_2$, so $\operatorname{edom}_1 \mathbb{r} = \mathbb{k} \times \mathbb{k}$. Therefore $\mathcal{D}(\mathbb{r}; 0) \neq \operatorname{edom} \mathbb{r}$.

Remark 2.2. [K-VV09, Theorem 3.1] fails for $r = (1 - x_1)x_2(1 - x_1)^{-1}$ because [K-VV09, Corollary 2.20] does not hold. Let $\mathcal{L}_2$ be the left shift operator on nc rational functions regular at the origin with respect to x_2; see [K-VV09, Subsection 2.2] for the definition. Then $\mathcal{L}_2(r) = (1 - x_1)^{-1}$ and hence $\operatorname{edom} \mathcal{L}_2(\mathbb{r}) \not\supseteq \operatorname{edom} \mathbb{r}$, contradicting [K-VV09, Corollary 2.20]. Furthermore, this lack of inclusion occurs because $\operatorname{edom} \mathbb{r}$ is not closed under direct sums: for example, $(1, 1), (0, 0) \in \operatorname{edom}_1 \mathbb{r}$ but $(1 \oplus 0, 1 \oplus 0) \notin \operatorname{edom}_2 \mathbb{r}$ and $(I_2, I_2) \notin \operatorname{edom}_2 \mathbb{r}$.

Remark 2.3. While the sets $\operatorname{dom} \mathbb{r}$ and $\operatorname{edom} \mathbb{r}$ are both closed under simultaneous conjugation, we have seen in Remark 2.2 that $\operatorname{edom} \mathbb{r}$ is not closed under direct sums. In terms of noncommutative function theory [K-VV14], the extended domain of a nc rational function $\mathbb{r}$ is thus not necessarily a noncommutative set and is therefore not a "natural" domain of the noncommutative function $\mathbb{r}$.

Moreover, since $\operatorname{edom} \mathbb{r} \neq \mathcal{D}(\mathbb{r}; 0)$ in general, it is a priori not even clear that $\operatorname{dom} \mathbb{r} \subseteq D(\mathbb{r}; 0)$. This could encumber operating with the domain of a nc rational function. Note that since the domain of a function is defined as the union of the domains of its representatives, determining whether a given tuple of matrices belong to the domain of $\mathbb{r}$ is not straightforward. In particular, we would not even know if $\operatorname{dom} \mathbb{r}$ is closed under direct sums, which could further compromise the study of nc rational functions from the free analysis perspective (note however that $\operatorname{dom} r$ is closed under direct sums for every $r \in \mathcal{R}_{\mathbb{k}}(\boldsymbol{x})$).

Remark 2.4. If $X \in \mathrm{M}_n(\mathbb{k})^g$ and $\ell \in \mathbb{N}$, then

$$I_\ell \otimes X = \overbrace{X \oplus \cdots \oplus X}^{\ell} \in \mathrm{M}_{\ell n}(\mathbb{k})^g$$

is an *ampliation* of X. As seen in Remark 2.2, $\operatorname{edom} \mathbb{r}$ is in general not even closed under ampliations. However, observe that $I_\ell \otimes X \in \operatorname{edom} \mathbb{r}$ for some $\ell \in \mathbb{N}$ implies $X \in \operatorname{edom} \mathbb{r}$.

3. Main results

In this section we introduce the notion of a stable extended domain of a nc rational function. Unlike the extended domain, the stable extended domain is closed under direct sums and Theorem 3.5 shows that it can be described by a monic linear pencil. In Theorem 3.10 we furthermore show that the stable extended domain and the domain coincide for nc rational functions regular at some scalar point.

3.1. Stable extended domain.

We start by defining the notion of a domain that will help us mend [K-VV09, Theorem 3.1].

Definition 3.1. Let $\mathbb{r} \in \mathbb{k}\langle\!\langle \boldsymbol{x} \rangle\!\rangle$. **The stable extended domain** of $\mathbb{r}$ is

$$\operatorname{edom}^{\rm st} \mathbb{r} = \bigcup_{n\in\mathbb{N}} \operatorname{edom}^{\rm st}_{\rm n} \mathbb{r},$$

where

$$\operatorname{edom}^{\rm st}_{\rm n} \mathbb{r} = \{X \in \operatorname{edom}_n \mathbb{r} \colon I_\ell \otimes X \in \operatorname{edom} \mathbb{r} \text{ for all } \ell \in \mathbb{N}\}.$$

Observe that $\operatorname{edom}^{\rm st} \mathbb{r}$ is closed under simultaneous conjugation and

$$\operatorname{dom} \mathbb{r} \subseteq \operatorname{edom}^{\rm st} \mathbb{r} \subseteq \operatorname{edom} \mathbb{r}.$$

Lemma 3.2. *Let $\mathbb{r} \in \mathbb{k}\langle\!\langle \boldsymbol{x} \rangle\!\rangle$ and assume $\operatorname{dom}_n \mathbb{r} \neq \emptyset$. Let Ξ be a tuple of $(2n) \times (2n)$ generic matrices and let $p = p(\Xi)$ be the least common multiple of the denominators of entries in $\mathbb{r}[2n]$. If Ξ' and Ξ'' are tuples of independent $n \times n$ generic matrices, then*

$$p(\Xi' \oplus \Xi'') = p_1(\Xi')p_2(\Xi'')$$

for some commutative polynomials p_1 and p_2.

Proof. Let r be a representative of $\mathbb{r}$ with $\operatorname{dom}_n r \neq \emptyset$. By [HMS+, Theorem 5.2] there exist $d \in \mathbb{N}$, $\mathbf{u}, \mathbf{v} \in \mathbb{k}^{d\times 1}$ and $Q = Q_0 + \sum_{j=1}^{g} Q_j x_j$ with $Q_j \in \mathbb{k}^{d\times d}$ such that

$$\mathbb{r} = \mathbf{v}^t Q^{-1} \mathbf{u}$$

holds in $\mathbb{k}\langle\!\langle \boldsymbol{x} \rangle\!\rangle$ and $Q(X)$ is invertible for every $X \in \operatorname{dom} r$. Since $\operatorname{dom}_n r \neq \emptyset$, $q = \det Q(\Xi)$ is a nonzero polynomial. If a is an invertible square matrix, then the entries of $(\det a)a^{-1}$ are polynomials in entries of a. Hence we conclude that p divides q. Observe that

$$q(\Xi' \oplus \Xi'') = \det Q(\Xi' \oplus \Xi'') = \det Q(\Xi') \det Q(\Xi'') = q_1(\Xi')q_2(\Xi'').$$

Since $p(\Xi' \oplus \Xi'')$ divides $q_1 q_2$, which is a product of polynomials in disjoint sets of variables, we see that $p(\Xi' \oplus \Xi'') = p_1 p_2$ for some p_1 dividing q_1 and p_2 dividing q_2. □

Proposition 3.3. *Every stable extended domain is closed under direct sums.*

Proof. If $Y' \in \mathrm{M}_{n'}(\mathbb{k})^g$ and $Y'' \in \mathrm{M}_{n''}(\mathbb{k})^g$, then $I_{n'n''} \otimes (I_\ell \otimes (Y' \oplus Y''))$ is after a canonical shuffle equal to $(I_{\ell n''} \otimes Y') \oplus (I_{\ell n'} \otimes Y'')$. By Remark 2.4 and Definition 3.1 it therefore suffices to prove the following claim: if $X', X'' \in \operatorname{edom}^{\rm st}_{\rm n} \mathbb{r}$, then $X' \oplus X'' \in \operatorname{edom}_{2n} \mathbb{r}$.

Let p be the least common multiple of the denominators of entries in $\mathbb{r}[2n]$. Then

$$\operatorname{edom}_{2n} \mathbb{r} = \{X \in \mathrm{M}_{2n}(\mathbb{k})^g \colon p(X) \neq 0\}.$$

Let Ξ' and Ξ'' be tuples of independent $n \times n$ generic matrices. By Lemma 3.2 we have

$$p(\Xi' \oplus \Xi'') = p_1(\Xi')p_2(\Xi'')$$

for some commutative polynomials p_1 and p_2. Hence

$$X' \oplus X'' \in \operatorname{edom}_{2n} \mathbb{r} \iff p_1(X')p_2(X'') \neq 0.$$

Since $X' \oplus X', X'' \oplus X'' \in \operatorname{edom}_{2n} \mathbb{r}$, we have

$$p_1(X')p_2(X') \neq 0, \qquad p_1(X'')p_2(X'') \neq 0$$

and thus $p_1(X')p_2(X'') \neq 0$, so $X' \oplus X'' \in \operatorname{edom}_{2n} \mathbb{r}$. □

3.2. **Corrigendum to the theorem of Kaliuzhnyi-Verbovetskyi and Vinnikov.** In this subsection we correct the proof and the statement of [K-VV09, Theorem 3.1]. The only gap in their ingenious proof appears at the very beginning in [K-VV09, Remark 2.7]. From the assumption that $\operatorname{edom} \mathbb{r}$ is closed under direct sums they derive that the left and right shifts of $\mathbb{r}$ do not shrink the domain in [K-VV09, Corollary 2.20]. Then they show in [K-VV09, Theorem 3.1] that if $(\mathbf{c}, L, \mathbf{b})$ is a minimal realization of $\mathbb{r}$, then every entry of L is a linear combination of iterated left and right shifts of $\mathbb{r}$. Combining these two results they deduce that $\operatorname{edom} \mathbb{r} = \mathcal{D}(\mathbb{r}; 0)$. We will show that these two main steps are indeed valid if $\operatorname{edom} \mathbb{r}$ is replaced by $\operatorname{edom}^{\mathrm{st}} \mathbb{r}$.

Let $\mathcal{L}_j$ be the left shift operator with respect to x_j as defined in [K-VV09, Subsection 2.2]. By [K-VV09, Remark 2.25], $\mathcal{L}_j$ can be described as follows. Every $\mathbb{r} \in \mathbb{k}\,\langle\!\!\langle \boldsymbol{x} \rangle\!\!\rangle$ that is regular at the origin can be expanded into a noncommutative power series about 0

$$\sum_{w \in <\boldsymbol{x}>} \alpha_w w;$$

then $\mathcal{L}_j(\mathbb{r}) \in \mathbb{k}\,\langle\!\!\langle \boldsymbol{x} \rangle\!\!\rangle$ is regular at 0 with the power series expansion

$$\sum_{w \in <\boldsymbol{x}>} \alpha_{x_j w} w.$$

Lemma 3.4. *If $0 \in \operatorname{dom} \mathbb{r}$, then $\operatorname{edom}^{\mathrm{st}} \mathbb{r} \subseteq \operatorname{edom}^{\mathrm{st}} \mathcal{L}_j(\mathbb{r})$ for $1 \le j \le g$.*

Proof. By Definition 3.1 it suffices to prove $\operatorname{edom}^{\mathrm{st}} \mathbb{r} \subseteq \operatorname{edom} \mathcal{L}_j(\mathbb{r})$. If $X \in \operatorname{edom}^{\mathrm{st}}_{\mathrm{n}} \mathbb{r}$, then $X \oplus 0 \in \operatorname{edom}_{n+1} \mathbb{r}$ by Proposition 3.3. If Ξ is a tuple of $n \times n$ generic matrices, $\boldsymbol{\xi} = (\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_g)$ a tuple of generic rows and ζ a commutative variable, then the properties of arithmetic operations on block lower triangular matrices imply that the denominators of the entries in

$$\mathbb{r}[n+1] \begin{pmatrix} \Xi & 0 \\ \boldsymbol{\xi} & \zeta \end{pmatrix}$$

are independent of variables in $\boldsymbol{\xi}$. Hence we have

$$\begin{pmatrix} X & 0 \\ \mathbf{w} & 0 \end{pmatrix} \in \operatorname{edom}_{n+1} \mathbb{r}$$

for all $\mathbf{w} \in (\mathbb{k}^{1\times n})^g$. Exactly as in the proof of [K-VV09, Corollary 2.20] we apply [K-VV09, Theorem 2.19] to show that X then belongs to the intersection of the domains of the denominators of entries in the row vector $\sum_j \mathbf{w}_j \mathcal{L}_j(\mathbb{r})[n]$ for every $\mathbf{w} \in (\mathbb{k}^{1\times n})^g$. In particular, by choosing $\mathbf{w}$ from the standard basis of $(\mathbb{k}^{1\times n})^g$ we conclude that $X \in \operatorname{edom} \mathcal{L}_j(\mathbb{r})$. □

Theorem 3.5. *Let $\mathbb{r} \in \mathbb{k}\,\langle\!\!\langle \boldsymbol{x} \rangle\!\!\rangle$. If $0 \in \operatorname{dom} \mathbb{r}$, then $\operatorname{edom}^{\mathrm{st}} \mathbb{r} = \mathcal{D}(\mathbb{r}; 0)$.*

Proof. Observe that $\mathcal{D}(\mathbb{r};0) \subseteq \operatorname{edom}^{\mathrm{st}} \mathbb{r}$ because $\mathcal{D}(\mathbb{r};0)$ is closed under direct sums. The converse inclusion is (mutatis mutandis) proved precisely in the same way as [K-VV09, Theorem 3.1], where we utilize Lemma 3.4 instead of [K-VV09, Corollary 2.20]. □

Remark 3.6. As mentioned in [K-VV09, Remark 3.2], the recognizable series realization implicitly appearing in Theorem 3.5 can be in fact replaced by a more general form of a state space realization, which also covers the noncommutative Fornasini-Marchesini realization [BGM05], the noncommutative Kaliuzhnyi-Verboverskyi realization [BK-V08] and the pure butterfly realization [HMV06].

Corollary 3.7. *Let* $\mathbb{r} \in \Bbbk\,\langle\!\!\langle \boldsymbol{x} \rangle\!\!\rangle$*. If* $\alpha \in \operatorname{dom}_1 \mathbb{r}$*, then* $\operatorname{edom}^{\mathrm{st}} \mathbb{r} = \mathcal{D}(\mathbb{r};\alpha)$*.*

Proof. (1) Let $\mathbb{r}_\alpha = \mathbb{r}(x+\alpha) \in \Bbbk\,\langle\!\!\langle \boldsymbol{x} \rangle\!\!\rangle$. Then

$$\operatorname{edom}^{\mathrm{st}}_{\mathrm{n}} \mathbb{r} = I_n\alpha + \operatorname{edom}^{\mathrm{st}}_{\mathrm{n}} \mathbb{r}_\alpha \qquad \text{and} \qquad \mathcal{D}(\mathbb{r};\alpha) = I_n\alpha + \mathcal{D}(\mathbb{r}_\alpha;0)$$

for all $n \in \mathbb{N}$. Now we apply Theorem 3.5 to $\mathbb{r}_\alpha$ to yield the conclusion. □

Remark 3.8. In [Vol+], matrix coefficient realization theory is applied to extend Corollary 3.7 to arbitrary nc rational functions (i.e., those not necessarily defined at a scalar point) in terms of their Sylvester realizations. Thus every stable extended domain can be described as the invertibility set of a generalized monic pencil; see [Vol+, Corollary 5.9] for the precise statement.

3.3. **Domains of functions regular at a scalar point.** In this subsection we improve Corollary 3.7 and precisely describe $\operatorname{dom} \mathbb{r}$ for a nc rational function $\mathbb{r}$ with $\operatorname{dom}_1 \mathbb{r} \neq \emptyset$. We require the following technical lemma.

Lemma 3.9. *Let* $\mathbb{m}$ *be a* $d \times d$ *matrix over* $\Bbbk\,\langle\!\!\langle \boldsymbol{x} \rangle\!\!\rangle$ *and let* $X \in \operatorname{dom} \mathbb{m}$*. If* $\det \mathbb{m}(X) \neq 0$*, then there exist nc rational expressions* s_{ij} *such that* $X \in \operatorname{dom} s_{ij}$ *and* $\mathbb{m}^{-1} = (\mathbb{s}_{ij})_{ij}$*.*

Proof. We prove the statement by induction on d. If $d = 1$, then $X \in \operatorname{dom} \mathbb{m}$ implies that there exists an expression $m \in \mathbb{m}$ with $X \in \operatorname{dom} m$. Since $\mathbb{m}(X) = m(X)$ is invertible, m^{-1} is the desired expression.

Now assume that the statement holds for $d-1$. Let m_{ij} be rational expressions satisfying $m_{ij} \in \mathbb{m}_{ij}$ and $X \in \operatorname{dom} m_{ij}$. Since $\mathbb{m}(X)$ is invertible, there exists a univariate polynomial $f \in \Bbbk[t]$ such that $f(\mathbb{m}(X))\mathbb{m}(X) = I$. Write $\widetilde{\mathbb{m}} = f(\mathbb{m})\mathbb{m}$. Let $u \in \widetilde{\mathbb{m}}_{11}$ be such that $X \in \operatorname{dom} u$. Since $\widetilde{\mathbb{m}}(X) = I$, we have $u(X) = I$ and consequently $X \in \operatorname{dom} u^{-1}$. As $\widetilde{\mathbb{m}}_{11}$ is a nonzero nc rational function, the Schur complement $\widehat{\mathbb{m}}$ of $\widetilde{\mathbb{m}}$ with respect to $\widetilde{\mathbb{m}}_{11}$ is well-defined. Note that $\widehat{\mathbb{m}}$ is a $(d-1) \times (d-1)$ matrix whose entries are products and sums of expressions m_{ij}, u^{-1}. Since $\widetilde{\mathbb{m}}(X)$ is invertible, $\widehat{\mathbb{m}}(X)$ is also invertible, so we can apply the induction hypothesis to $\widehat{\mathbb{m}}$. Hence there exist rational expressions s'_{ij} such that $X \in \operatorname{dom} s'_{ij}$ and $\widehat{\mathbb{m}}^{-1} = (\mathbb{s}'_{ij})_{ij}$. The entries of $\widetilde{\mathbb{m}}^{-1}$ can be represented by sums and products of expressions m_{ij}, u^{-1}, s'_{ij}, hence the same holds for $\mathbb{m}^{-1} = \widetilde{\mathbb{m}}^{-1} f(\mathbb{m})$. □

Theorem 3.10. *If* $\operatorname{dom}_1 \mathbb{r} \neq \emptyset$*, then* $\operatorname{dom} \mathbb{r} = \operatorname{edom}^{\mathrm{st}} \mathbb{r} = \mathcal{D}(\mathbb{r};\alpha)$ *for every* $\alpha \in \operatorname{dom}_1 \mathbb{r}$*.*

Proof. By Corollary 3.7 we have $\operatorname{dom} \mathbb{r} \subseteq \operatorname{edom}^{\mathrm{st}} \mathbb{r} = \mathcal{D}(\mathbb{r};\alpha)$. On the other hand, let $\mathbf{c}^t L^{-1} \mathbf{b}$ be a minimal realization of $\mathbb{r}$ about α and $X \in \mathcal{D}(\mathbb{r};\alpha)$. Hence $\det L(X) \neq 0$, so by Lemma 3.9 there exist nc rational expressions s_{ij} such that $X \in \operatorname{dom} s_{ij}$ and $L^{-1} = (\mathbb{s}_{ij})_{ij}$. Therefore $\mathbb{r}$ admits a representative

$$r = \sum_{ij} c_i s_{ij} b_j$$

with $X \in \operatorname{dom} r \subseteq \operatorname{dom} \mathfrak{r}$. □

Remark 3.11. Theorem 3.10 can be compared with [HM14], where authors consider *limit domains* and *hidden singularities* of symmetric nc rational functions over $\mathbb{R}$ that are regular at the origin. While the limit domain of a nc rational function might strictly contain its extended domain, we still observe the following analogy. If $\mathfrak{r} \in \Bbbk \mathopen{<\!\!\!(} \boldsymbol{x} \mathclose{)\!\!\!>}$ and $0 \in \operatorname{dom} \mathfrak{r}$, then [HM14, Theorem 1.5] roughly corresponds to $\operatorname{dom} \mathfrak{r} \cap (\operatorname{edom} \mathfrak{r} \setminus \mathcal{D}(\mathfrak{r}; 0)) = \emptyset$, i.e., $\operatorname{dom} \mathfrak{r} \subseteq \mathcal{D}(\mathfrak{r}; 0)$.

Remark 3.12. Nc rational expressions as defined in Subsection 2.1 are sometimes also called *scalar* nc rational expressions. On the other hand, a *matrix nc rational expression* is a syntactically valid combination of matrices over scalar nc rational expressions and matrix operations $+, \cdot, {}^{-1}$. Here we assume that the sum and the product are applied just to matrices of matching dimensions and the inverse is applied just to square matrices. Hence a nc rational function can be also represented by a 1×1 matrix nc rational expression. For example, if $\operatorname{dom}_1 \mathfrak{r} \neq \emptyset$, then any realization of $\mathfrak{r}$ can be viewed as a 1×1 matrix representative of $\mathfrak{r}$. Now Lemma 3.9 implies that any point $X \in \mathrm{M}_n(\Bbbk)^g$, at which some 1×1 matrix representative of $\mathfrak{r}$ is defined, belongs to $\operatorname{dom} \mathfrak{r}$ (which is defined only with scalar representatives).

3.4. **Absence of distinguished representatives.** Recall that the domain of a nc rational function is defined as the union of domains of its representatives. If $\mathfrak{r} \in \Bbbk \mathopen{<\!\!\!(} \boldsymbol{x} \mathclose{)\!\!\!>}$ and $r \in \mathfrak{r}$, then we say that r is a *distinguished representative* of $\mathfrak{r}$ if $\operatorname{dom} \mathfrak{r} = \operatorname{dom} r$. For example, if $\mathfrak{r}$ is the inverse of $f \in \Bbbk\langle \boldsymbol{x} \rangle \setminus \{0\}$, then $\operatorname{dom} \mathfrak{r} = \operatorname{edom} \mathfrak{r} = \operatorname{dom} f^{-1}$, where f^{-1} is naturally considered as an element of $\mathcal{R}_{\Bbbk}(\boldsymbol{x})$. While the existence of distinguished representatives may seem exceptional, up to this point there was no reported instance of a function without distinguished representatives. We present such a function in the following example.

Example 3.13. Let $g = 4$ and

$$\mathfrak{r} = \left(x_4 - x_3 x_1^{-1} x_2\right)^{-1} \in \Bbbk \mathopen{<\!\!\!(} \boldsymbol{x} \mathclose{)\!\!\!>}, \qquad m = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in \Bbbk\langle \boldsymbol{x} \rangle^{2\times 2}.$$

We claim that $\mathfrak{r}$ does not admit distinguished representatives. Since $(1,0,0,1) \in \operatorname{dom}_1 \mathfrak{r}$, we can use the minimal realization

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1+x_1 & x_2 \\ x_3 & 1+x_4 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

for $\mathfrak{r}(x_1+1, x_2, x_3, x_4+1)$ and Theorem 3.10 to show that

$$\operatorname{dom} \mathfrak{r} = \{X \colon \det m(X) \neq 0\}.$$

Suppose there exist $r \in \mathfrak{r}$ with $\operatorname{dom} r = \operatorname{dom} \mathfrak{r}$. Then r contains at least one inverse. Looking at the innermost nested inverse we conclude that there exists $f \in \Bbbk\langle \boldsymbol{x} \rangle$ of degree $d > 0$ such that

$$\det f(X) = 0 \quad \Rightarrow \quad \det m(X) = 0 \qquad \forall X \in \mathrm{M}_n(\Bbbk)^4,\ n \in \mathbb{N}. \tag{3.1}$$

Note that $f(x_2, x_1, x_4, x_3)$, $f(x_3, x_4, x_1, x_2)$ and $f(x_4, x_3, x_2, x_1)$ also satisfy (3.1), so we can assume that f contains a monomial $x_1 u_0$ with $u_0 \in \langle \boldsymbol{x} \rangle$ and $|u_0| = d - 1$. Let $u_1, \dots, u_M \in \langle \boldsymbol{x} \rangle$ be all the words of length at most $d-1$ except u_0 and let $w_1, \dots, w_N \in$

$<\boldsymbol{x}>$ be all the words of length d except x_1u_0; hence $M=\frac{4^d-1}{3}-1$ and $N=4^d-1$. Let $\mathcal{V}\subset \Bbbk<\boldsymbol{x}>$ be a $\Bbbk$-subspace spanned by $u_0,\dots,u_M,w_1,\dots,w_N$. Then $f=x_1u_0-h$ for $h\in\mathcal{V}$. Furthermore, for $k\in\{2,3\}$ choose some ordering of

$$\{v_1^{(k)},\dots,v_N^{(k)}\}=\{u_0,\dots,u_M,w_1,\dots,w_N\}\setminus x_k<\boldsymbol{x}>.$$

For $k\in\{1,2,3,4\}$ let $X_k:\mathcal{V}\to\mathcal{V}$ be linear maps defined by

$$\begin{aligned} X_ku_0&=\begin{cases} h & k=1\\ x_ku_0 & k>1\end{cases},\\ X_ku_i&=x_ku_i \qquad 1\le i\le M,\\ X_kw_j&=\begin{cases} 0 & k\in\{1,4\}\\ v_j^{(k)} & k\in\{2,3\}\end{cases}. \end{aligned}$$

Let $X=(X_1,X_2,X_3,X_4)$. By definition, we have

$$f(X)1=h-h=0.$$

On the other hand, if we identify $\mathcal{V}$ with $\Bbbk^{1+M+N}$ and the words $u_0,\dots,u_M,w_1,\dots,w_N$ with the standard basis vectors in $\Bbbk^{1+M+N}$, then the columns of the block matrix

$$\begin{pmatrix} X_1 & X_2\\ X_3 & X_4\end{pmatrix}=\begin{pmatrix} h & x_1u_1 & \cdots & x_1u_M & 0 & \cdots & 0 & x_2u_0 & x_2u_1 & \cdots & x_2u_M & v_1^{(2)} & \cdots & v_N^{(2)}\\ x_3u_0 & x_3u_1 & \cdots & x_3u_M & v_1^{(3)} & \cdots & v_N^{(3)} & x_4u_0 & x_4u_1 & \cdots & x_4u_M & 0 & \cdots & 0\end{pmatrix}$$

are independent by the construction. Therefore $f(X)$ is singular and $m(X)$ is invertible, a contradiction.

Note that, due to Remark 3.12 and Theorem 3.10, one can still say that a noncommutative rational function regular at a scalar point admits a distinguished 1×1 matrix representative, namely its minimal realization.

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